Uncertainty, information, and entropy: comparison of two definitions

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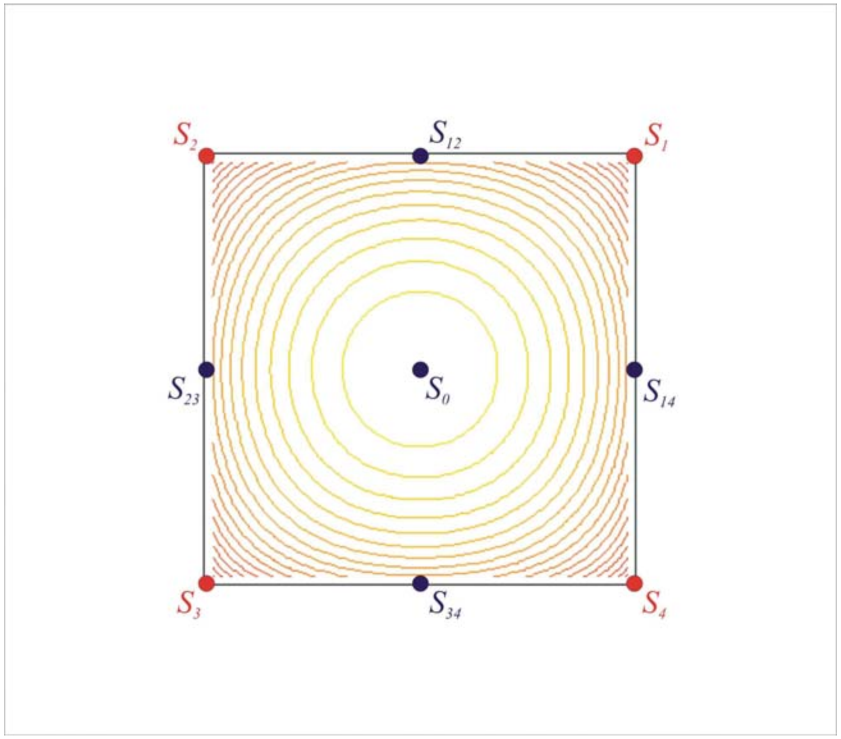

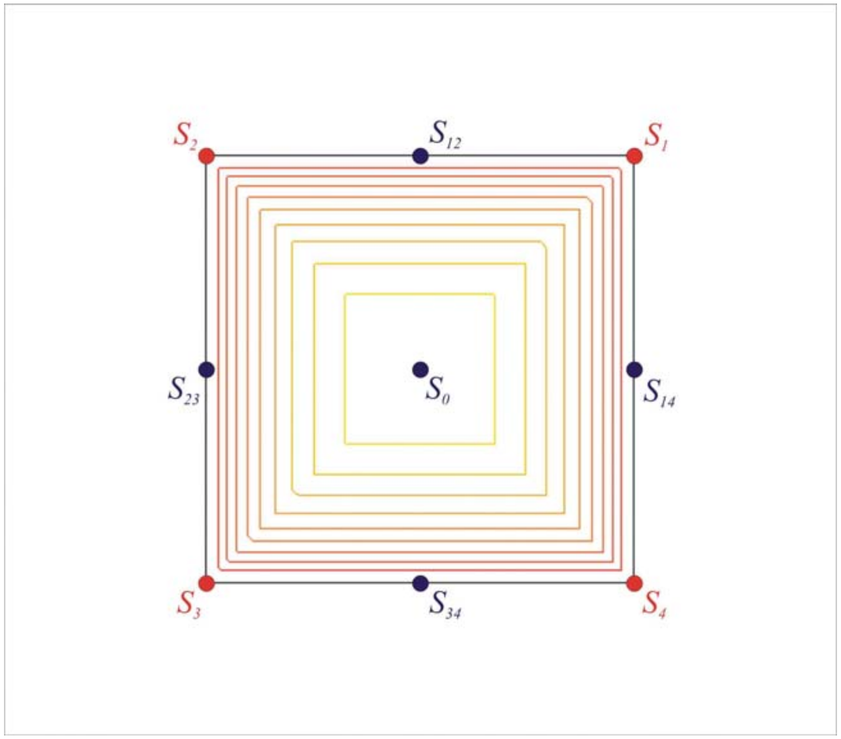

Outline

Entropy of a quantum state: applications and justifications

Analysis of two derivations: Blankenbecler & Partovi's and von Neumann's

The two definitions are appealing and agree for quantum systems

How sensible are they? ←→ Do they work for general statistical systems?

Application to a "toy model":
The two definitions do not agree. Problem with BP's definition

Conclusions

Entropy for quantum-system states

Quantum state retrodiction

Statistical mechanics

$$H(\rho) = -\mathrm{tr}(\rho \ln \rho)$$

Quantum entropy

Quantum state estimation

Equilibrium & non-equilibrium thermodynamics

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Entropy for quantum-system states: justifications and derivations

$$H(\rho) = -\mathrm{tr}(\rho \ln \rho)$$

Quantum entropy

"It works!"-arguments

Ochs
invariance arguments

von Neumann
thermodynamic arguments

Balian
analogies to Bayesian arguments

Blankenbecler and Partovi
intuitive information-theoretical arguments

Is a justification through purely probability-theoretical arguments possible? $\approx$

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Blankenbecler & Partovi's and von Neumann's definitions

$$H(\rho) = -\mathrm{tr}(\rho \ln \rho)$$

Blankenbecler and Partovi:

"the lack of information must be gauged against the most accurate measuring device available"

(Phys. Rev. Lett. **54** (1985) 373).

von Neumann:

Implementation of measurements through semi-permeable membranes → entropy through thermodynamic arguments

(*Mathematische Grundlagen der Quantenmechanik* (1932)).

Blankenbecler & Partovi's and von Neumann's definitions

The definitions by Blankenbecler & Partovi's and by von Neumann's seem appealing to us, but they are still based on intuitive arguments.

It can be useful to study how sensible they are by checking whether these definitions work for more general statistical systems.

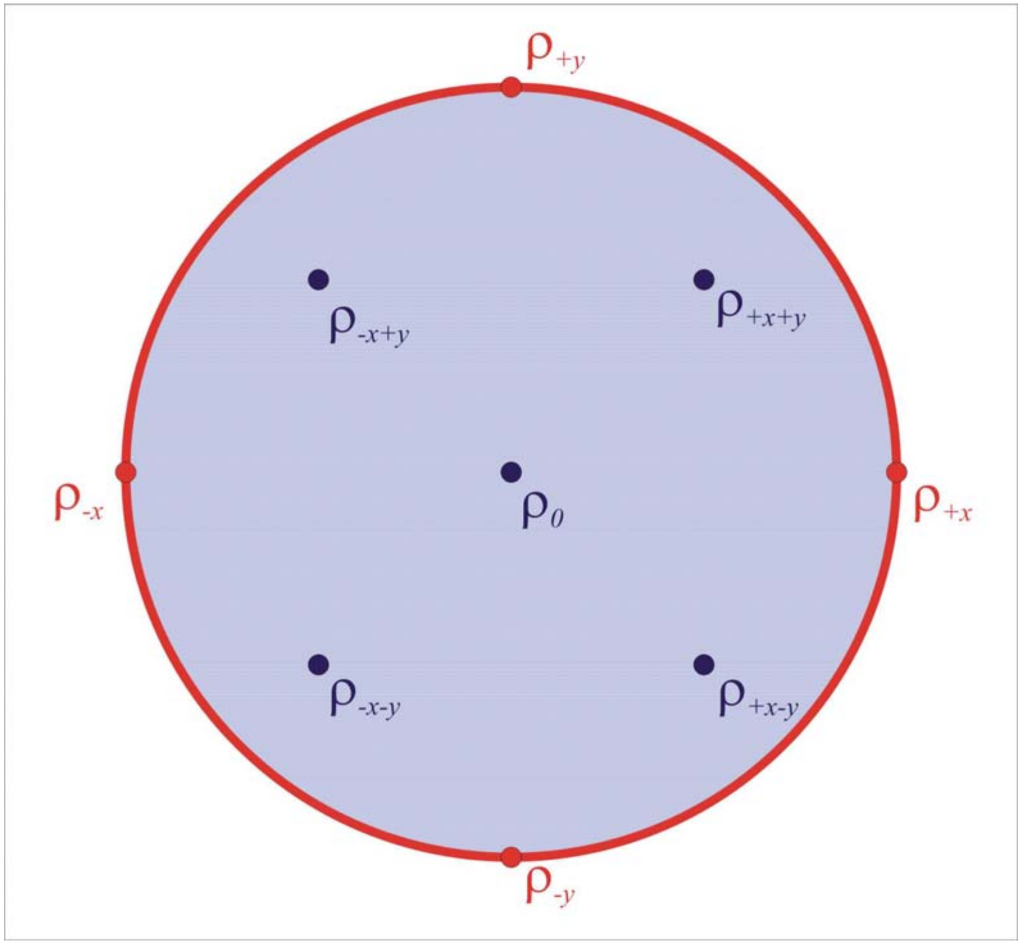
We begin by a review of how they work for quantum systems.

Two-level quantum system: statistical properties

		ρ_{+x}	ρ_{-x}	ρ_{+y}	ρ_{-y}	ρ_0	ρ_{+x+y}	ρ_{+x-y}	ρ_{-x+y}	ρ_{-x-y}	…
Π_x	P(+x)	1	0	1/2	1/2	1/2	3/4	3/4	1/4	1/4	…
	P(-x)	0	1	1/2	1/2	1/2	1/4	1/4	3/4	3/4	…
Π_y	P(+y)	1/2	1/2	1	0	1/2	3/4	1/4	3/4	1/4	…
	P(-y)	1/2	1/2	0	1	1/2	1/4	3/4	1/4	3/4	…
…	…	…	…	…	…	…	…	…	…	…	…

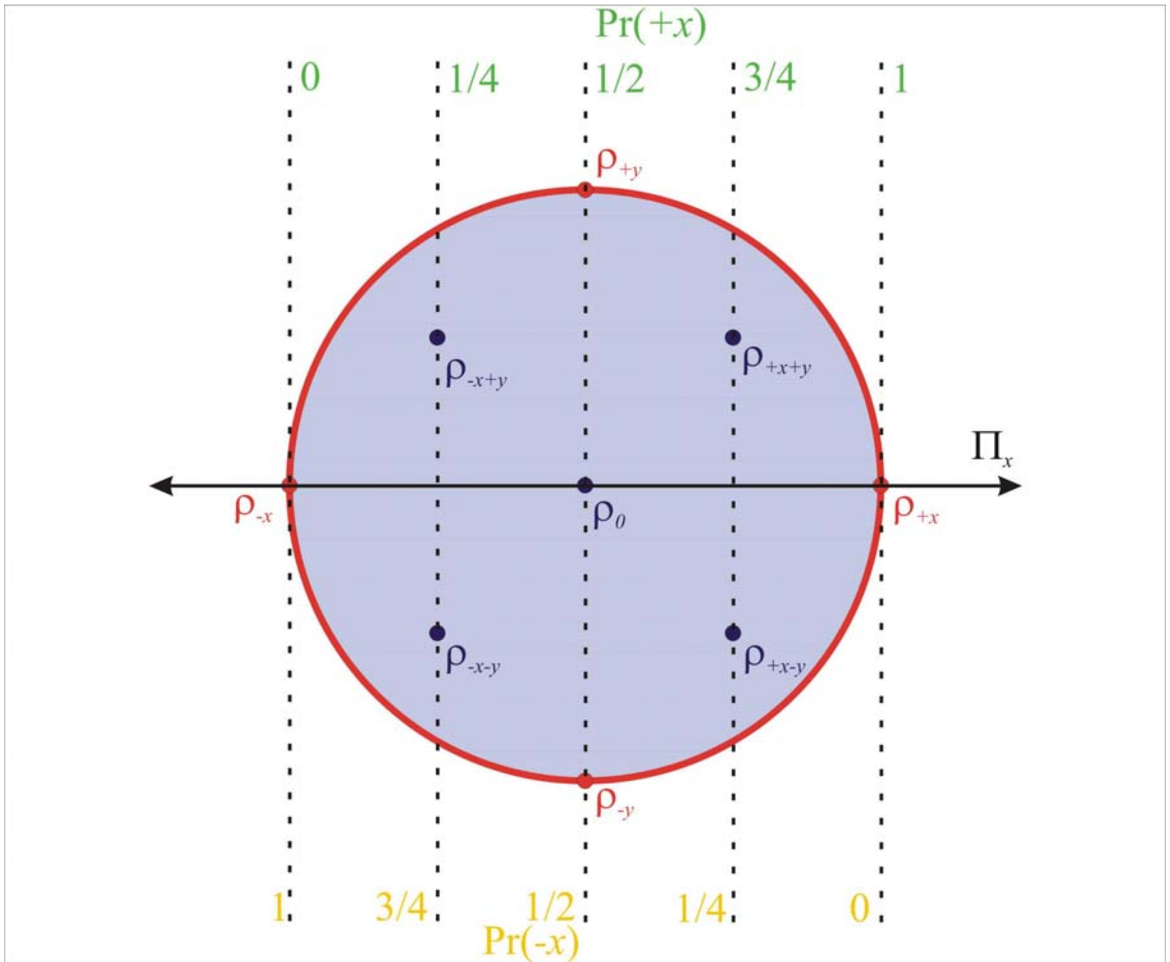
- Infinite number of pure states and ”maximal” measurements.
- Other states and measurements obtained by mixing or ”coarse-graining”.

Two-level quantum system: Bloch-sphere representation



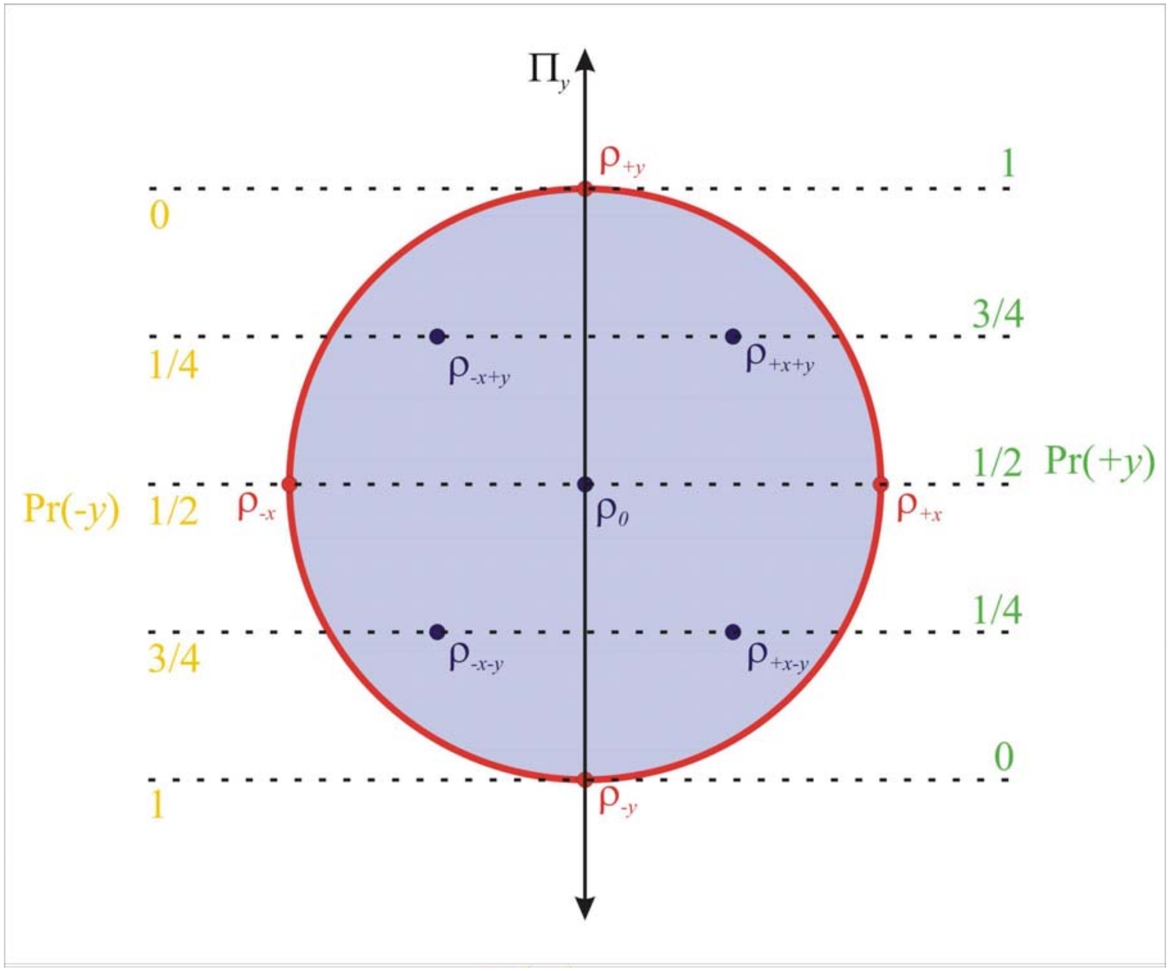
- The set of states can be represented as a sphere (convex properties).
- Infinite pure states (cannot be statistically simulated).
- All mixed states can be realised in more than one way.

Two-level quantum system: Bloch-sphere representation



- Measurements represented by isoprobability lines.
- Infinite number of maximal measurements.
- Uncertainty relations for some maximal measurements (no dispersion-free states).

Two-level quantum system: Bloch-sphere representation



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Blankenbecler & Partovi’s definition

Consider a state (density matrix)

ρ

Check the (Shannon) entropies for all (pure) measurements upon that state

$$H(\rho, \{\Pi_i\}) = -\Sigma_i \, \mathrm{tr}(\rho \, \Pi_i) \ln \mathrm{tr}(\rho \, \Pi_i)$$

Select the least of these entropies as the entropy of that state

$$H_{\mathrm{Par}}(\rho) = \min_{\{\Pi i\}} [H(\rho, \{\Pi_i\})] = -\mathrm{tr}(\rho \ln \rho)$$

von Neumann's definition

Imaginary ensemble of $N \to \infty$ systems/"particles" in a given state ("gas")

$$\rho \quad (N)$$

Implementation of measurements through semi-permeable membranes → Maximal reversible, isothermal performance of work to separate pure states

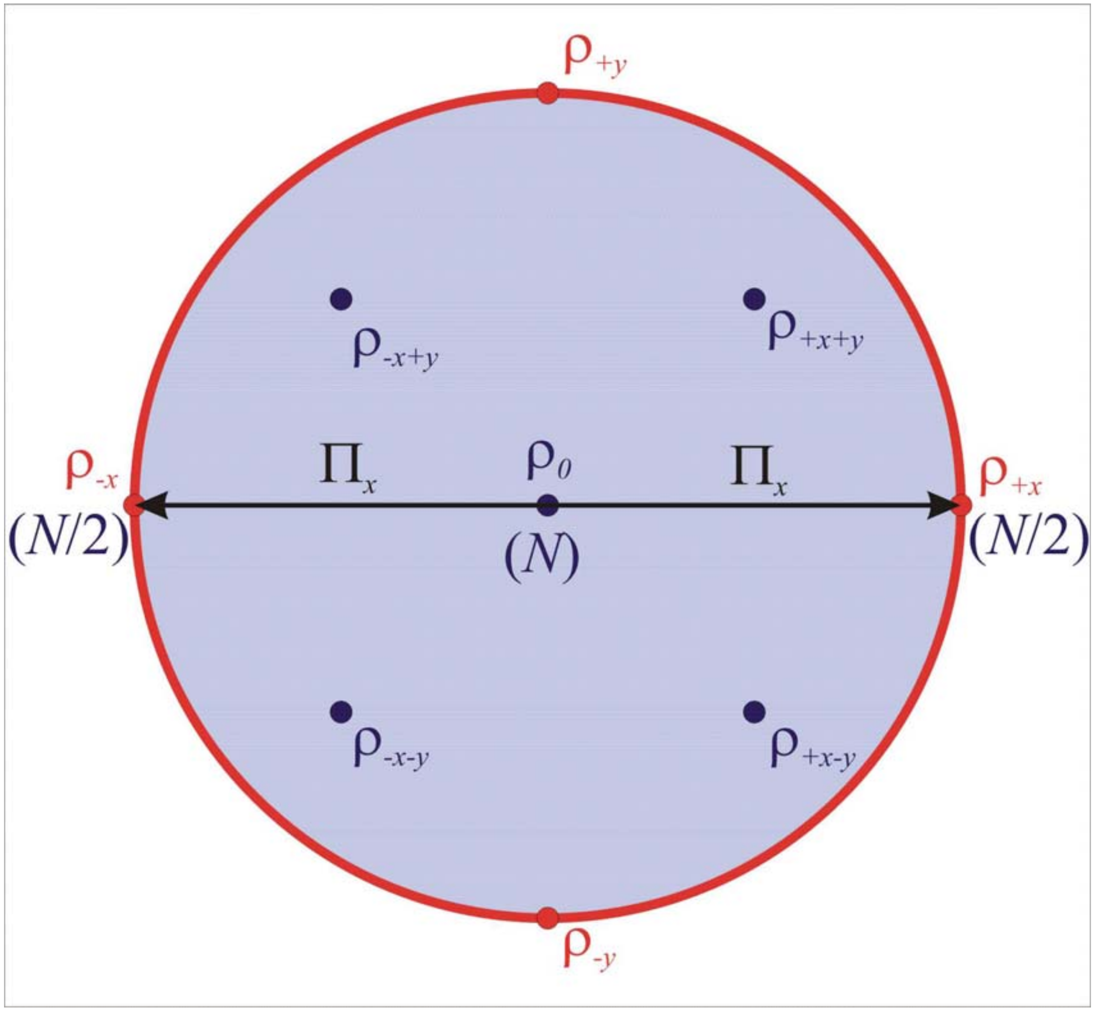
$$W = -NT \ln(V_f/V_i) = Q$$

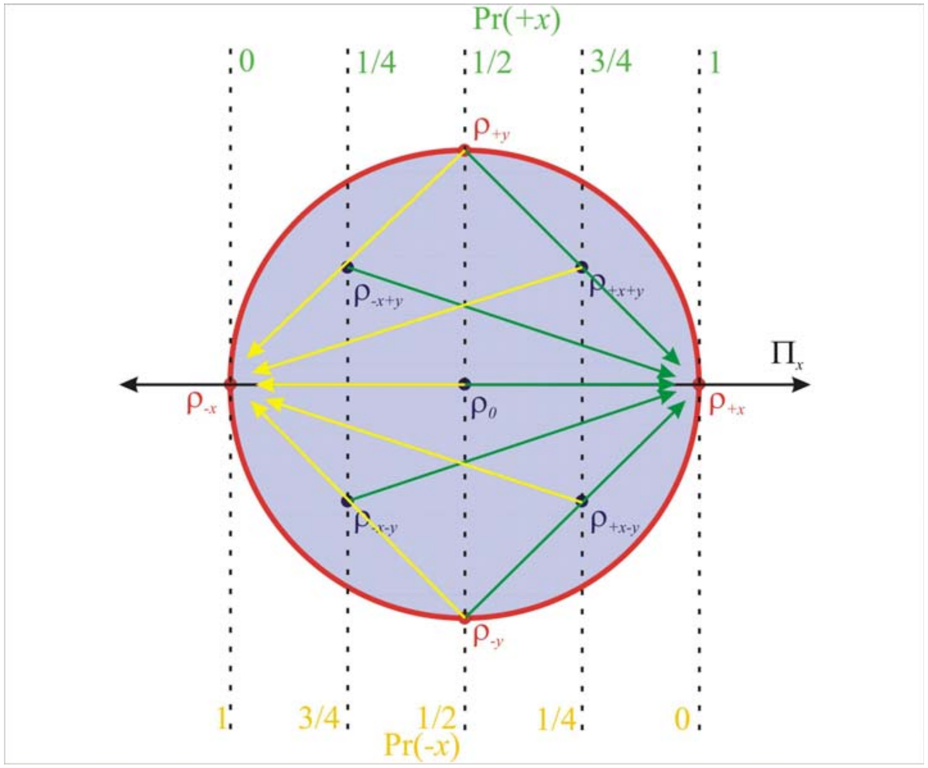
$(k=1)$

Thermodynamical entropy per "particle" (entropy of pure state = 0)

$$H_{vN}(\rho) = -\Delta S/N = \ln(V_f/V_i) = -\mathrm{tr}(\rho \ln \rho)$$

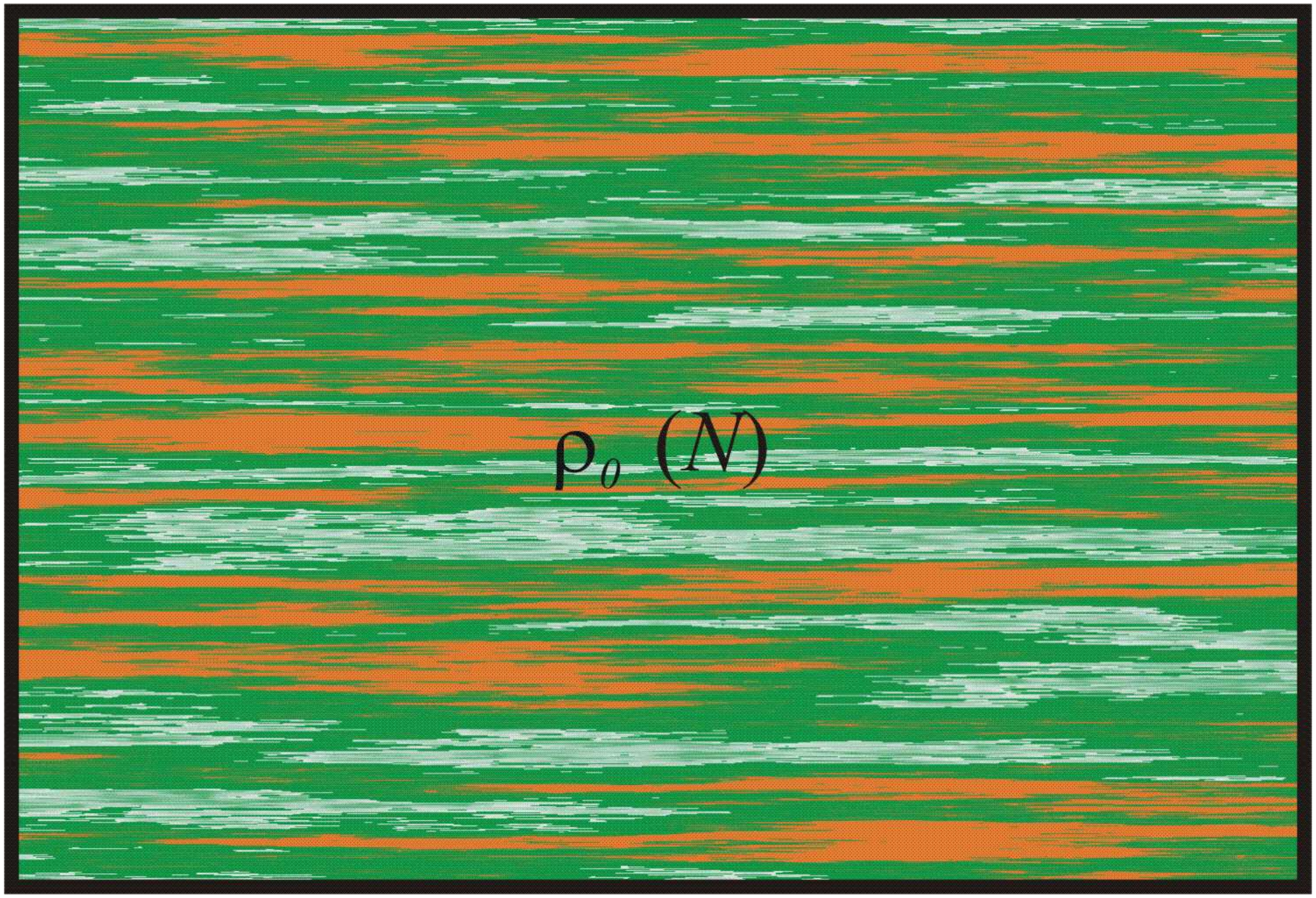
Example: vN entropy for the maximally unpolarised state $\rho_0 = I/2$





Implementation of the measurement Π_x

Example: vN entropy for the maximally unpolarised state $\rho_0 = I/2$



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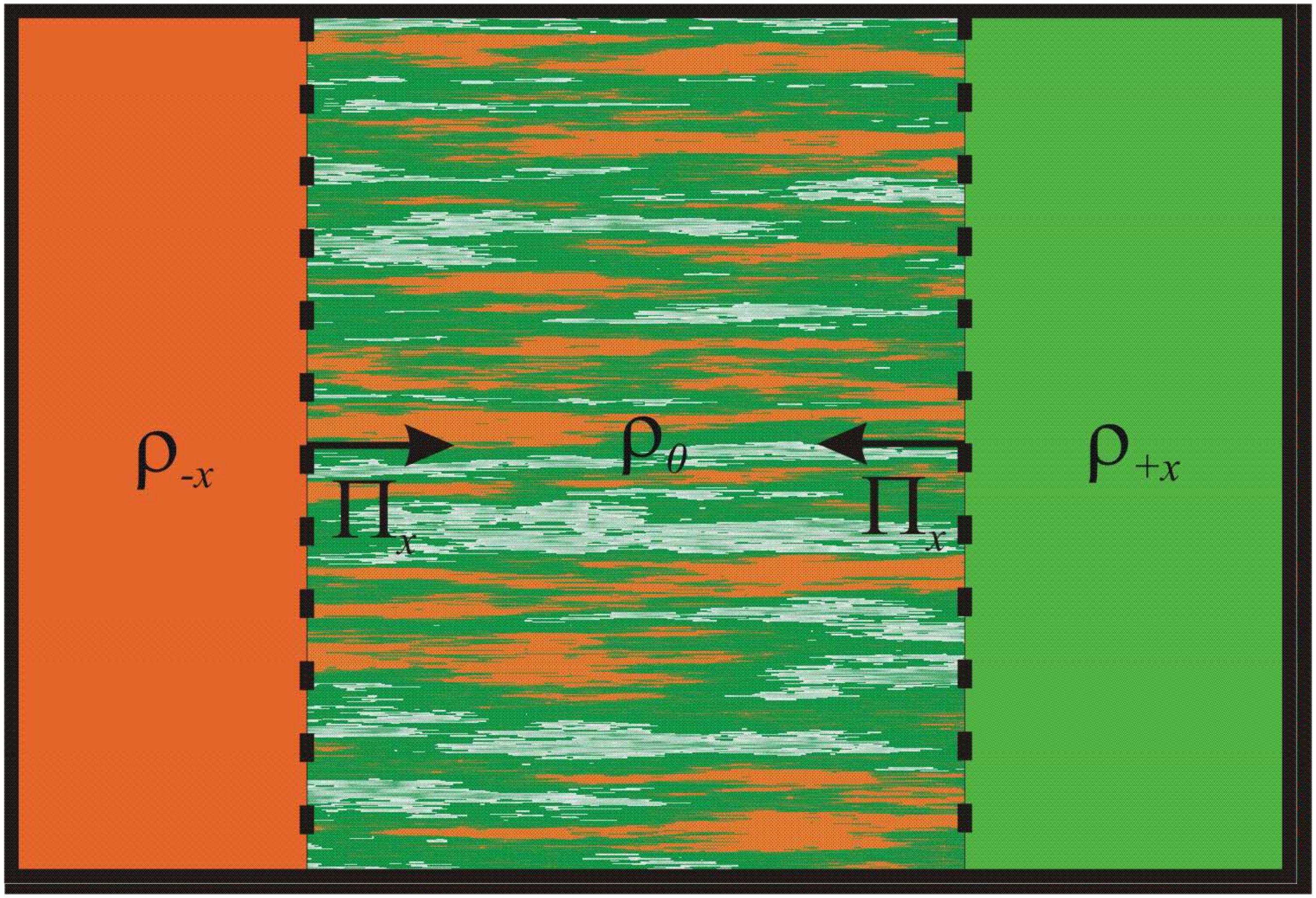

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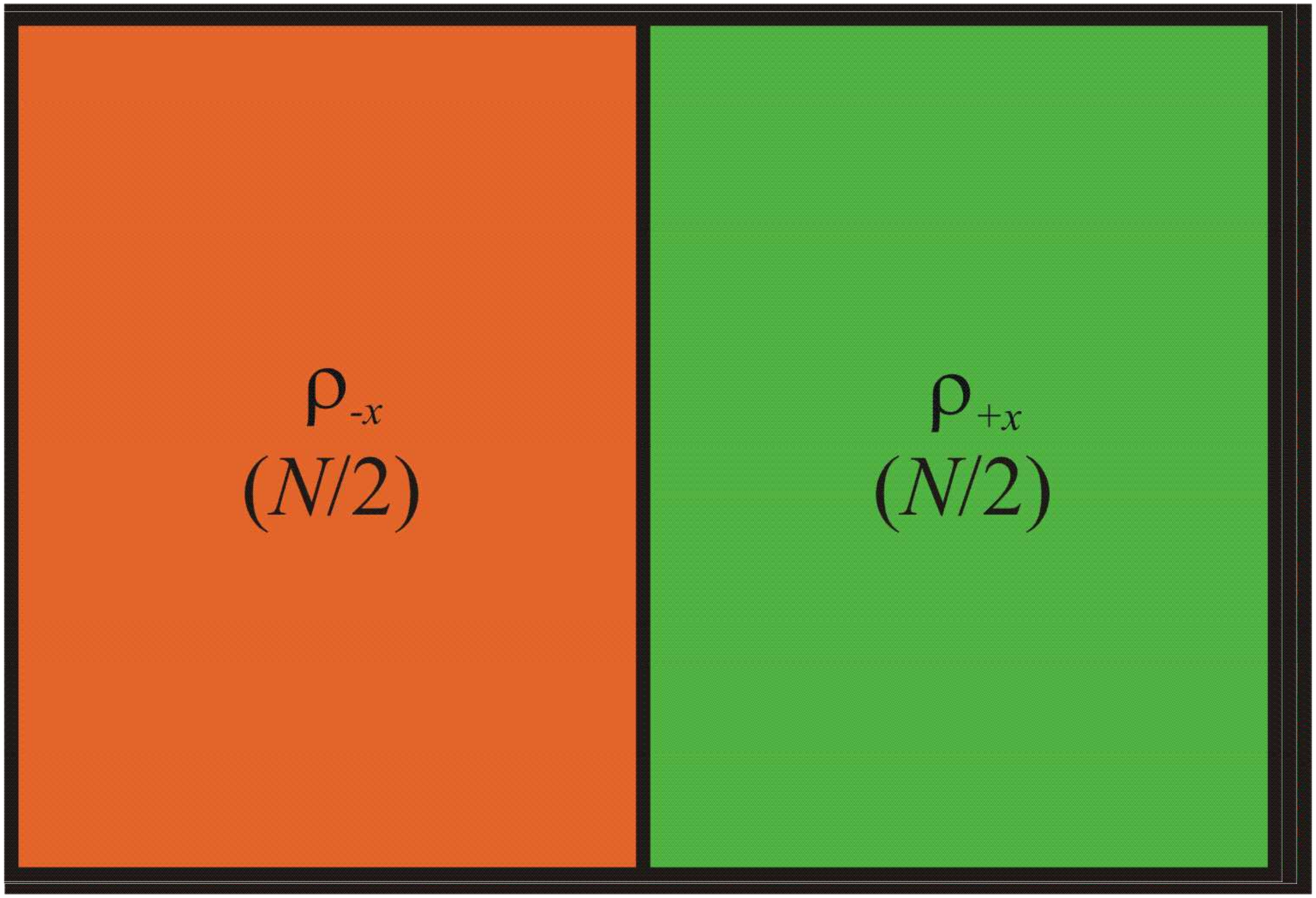

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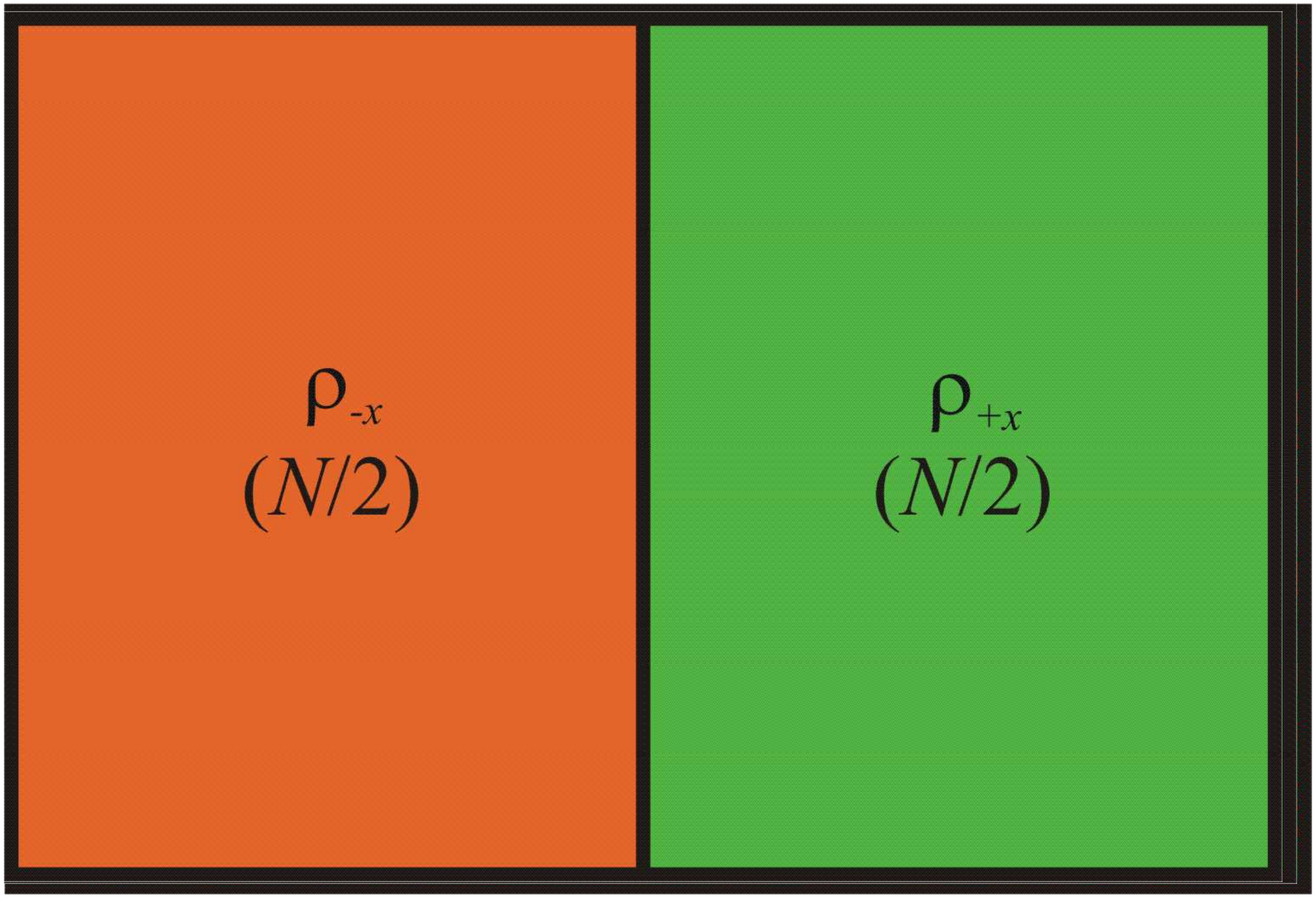
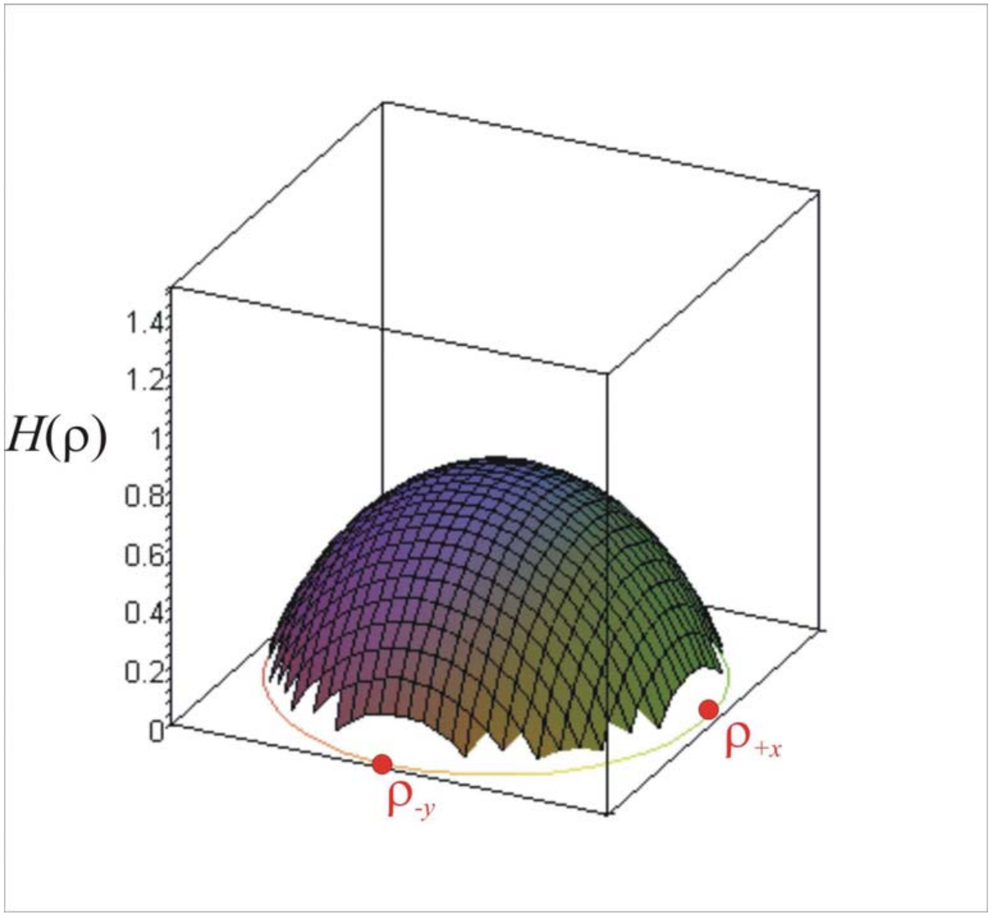
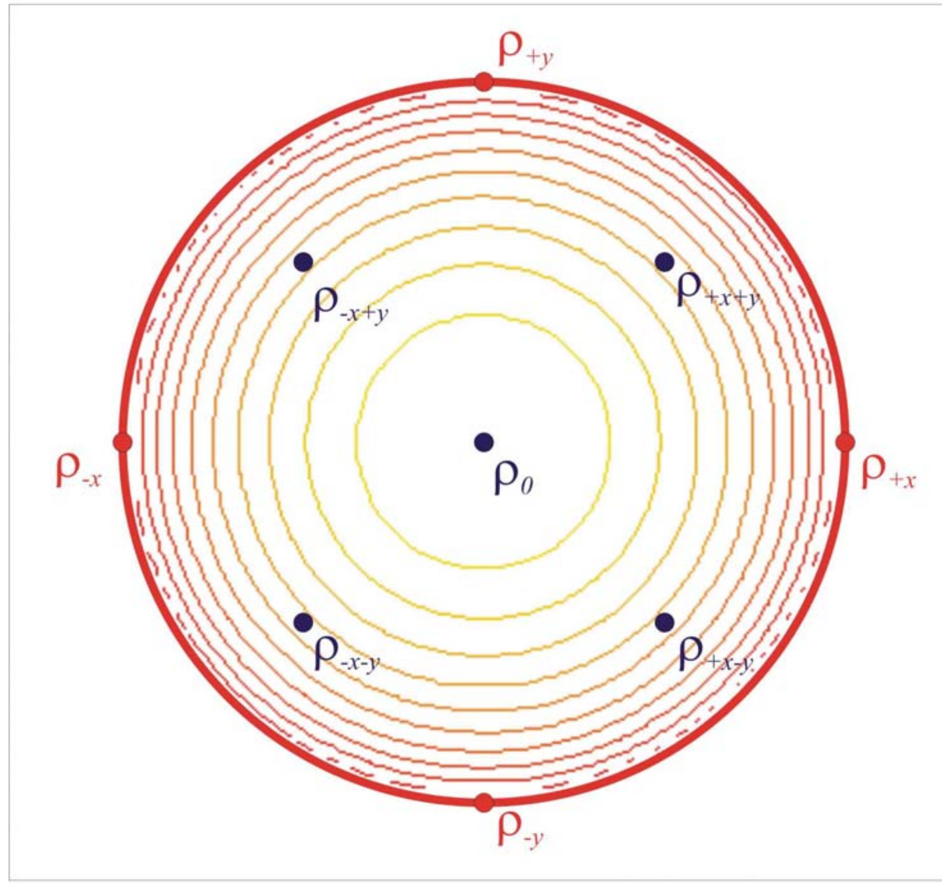


$$W = -2 \times N/2\ T \ln(1/2) = Q \quad \rightarrow \quad H_{\mathrm{vN}}(\rho_0) = \ln(2)$$

The two definitions lead to the same entropy formula

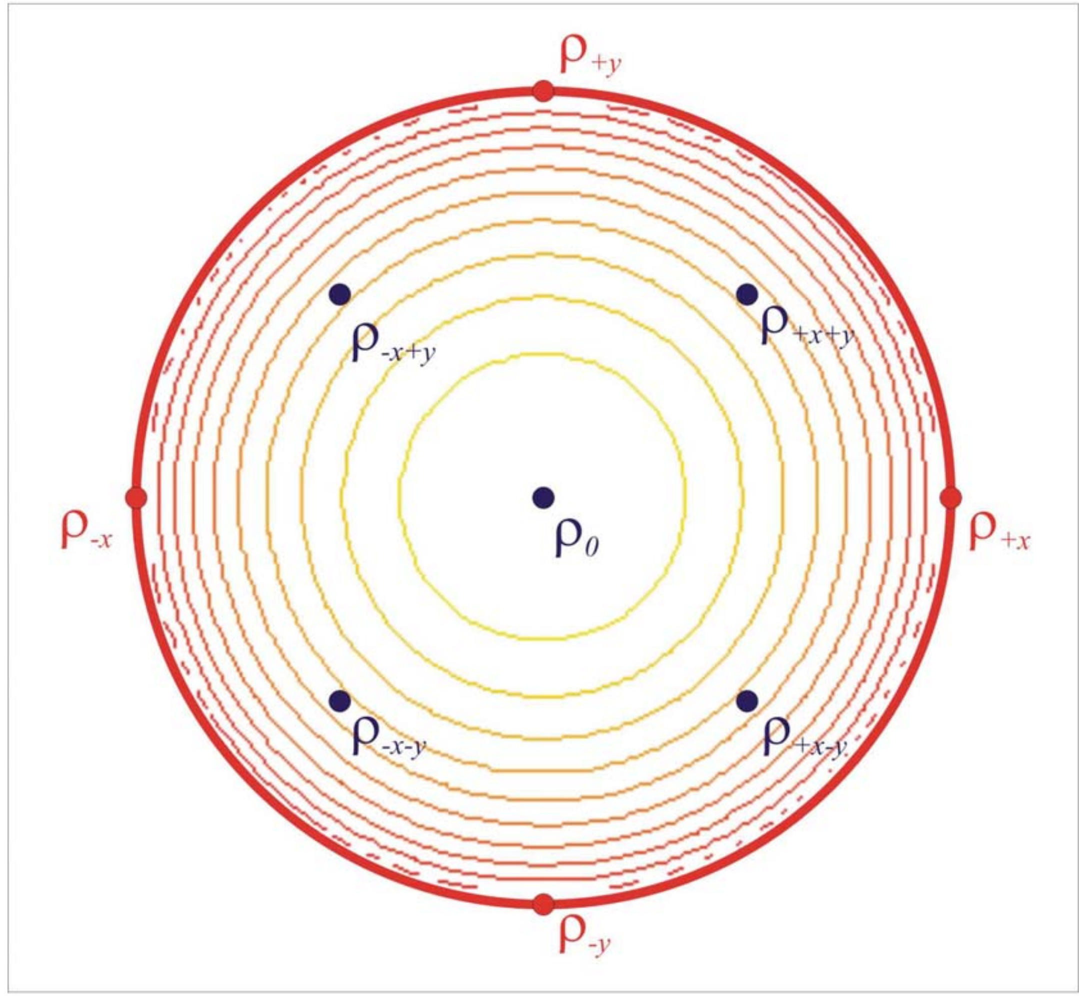




$$H_{\mathrm{Par}}(\rho) = H_{\mathrm{vN}}(\rho) = H(\rho) = -\mathrm{tr}(\rho \ln \rho)$$

von Neumann’s entropy definition applied to the toy model

The entropy which arises from the two entropy definitions is convex → it can be used in a constrained maximum-entropy principle



$$H_{vN}(\rho) = -\mathrm{tr}(\rho \ln \rho)$$

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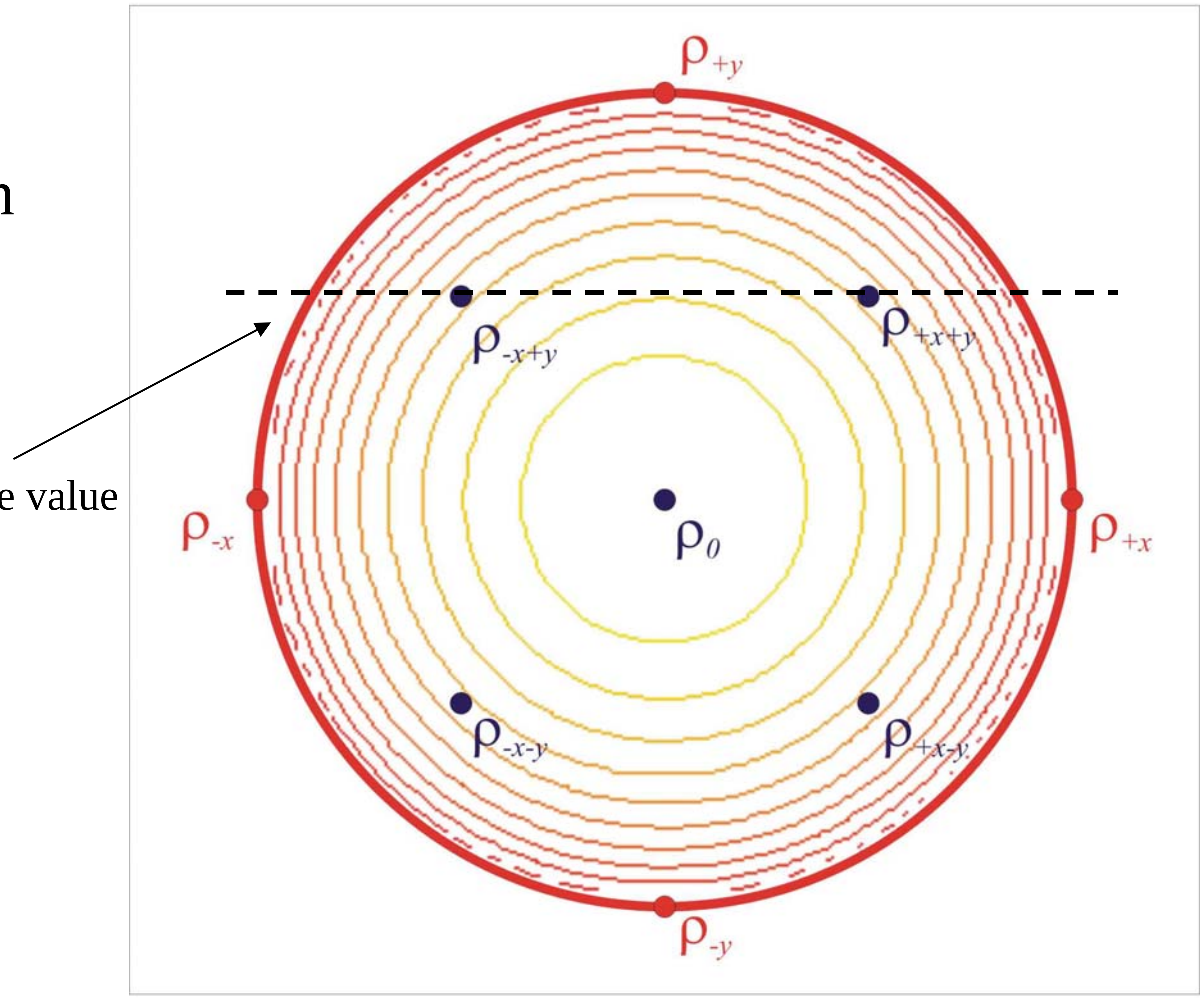
average value



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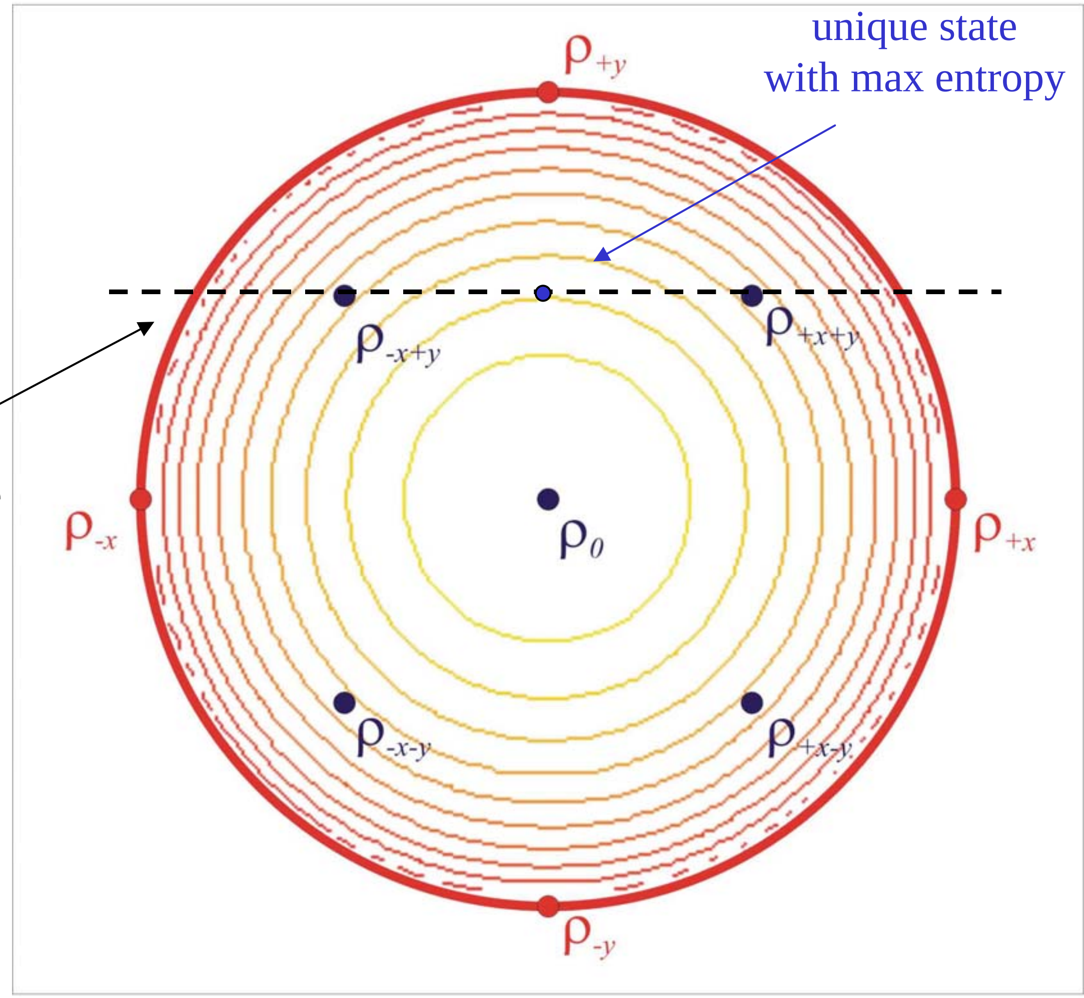

$H_{vN}(\rho) = -\mathrm{tr}(\rho \ln\rho)$

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Testing the two definitions with other kinds of statistical systems

- Are the two definitions equivalent for other kinds of statistical systems as well?
- Would they work in e.g. a maximum-entropy principle?

Blankenbecler and Partovi:

"the lack of information must be gauged against the most accurate measuring device available"

(Phys. Rev. Lett. **54** (1985) 373).

von Neumann:

Implementation of measurements through semi-permeable membranes → entropy through thermodynamic arguments

(*Mathematische Grundlagen der Quantenmechanik* (1932)).

$$\boxed{H_{\text{Par}}} \overset{?}{=} \boxed{H_{\text{vN}}}$$

Are the two definitions equivalent for other kinds of statistical systems as well?

- Statistically classical systems (simplexes): **Yes**

- Other statistical systems: **Not necessarily**

(e.g.: subsystems of quantum or classical systems with physical constraints)

Example: a "toy model"

- The model is not quantum-mechanical, but has some quantum properties (superposition, no dispersion-free states)

- It could arise as subsystem of a classical or quantum system because of symmetries or superselection rules

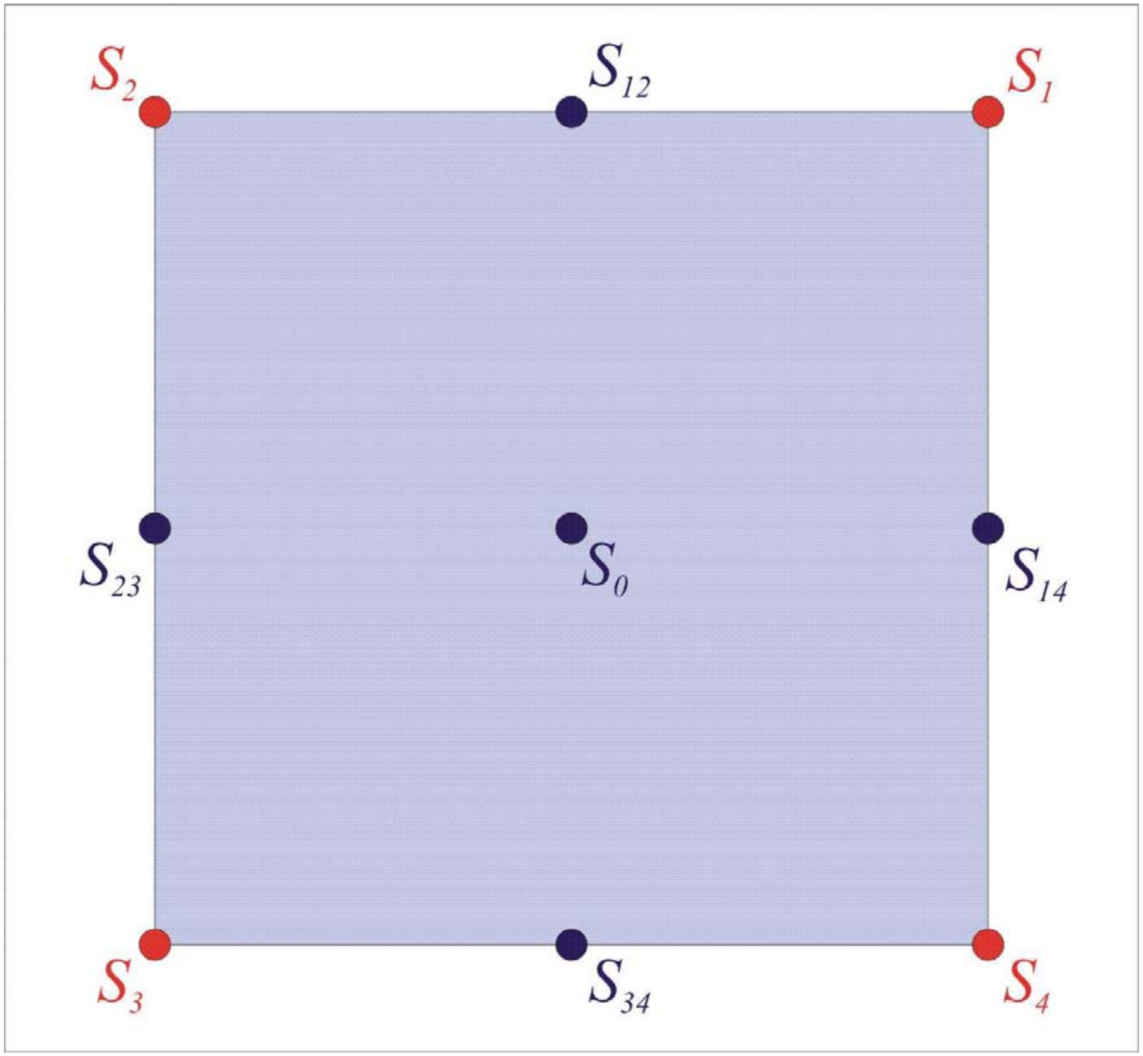
(Holevo: *Probabilistic and Statistical Aspects of Quantum Theory* (1982), § I.5)

Toy model: statistical properties (the rules of the game)

		S_1	S_2	S_3	S_4	S_0	S_{12}	S_{23}	S_{34}	S_{14}	…
M_y	$\mathrm{P}(R_1)$	1	1	0	0	1/2	1	1/2	0	1/2	…
	$\mathrm{P}(R_2)$	0	0	1	1	1/2	0	1/2	1	1/2	…
M_x	$\mathrm{P}(R_3)$	1	0	0	1	1/2	1/2	0	1/2	1	…
	$\mathrm{P}(R_4)$	0	1	1	0	1/2	1/2	1	1/2	0	…
…	…	…	…	…	…	…	…	…	…	…	…

- Four pure states and two (binary) ”maximal” measurements.
- Other states and measurements obtained by mixing or ”coarse-graining”.

Toy model: square representation

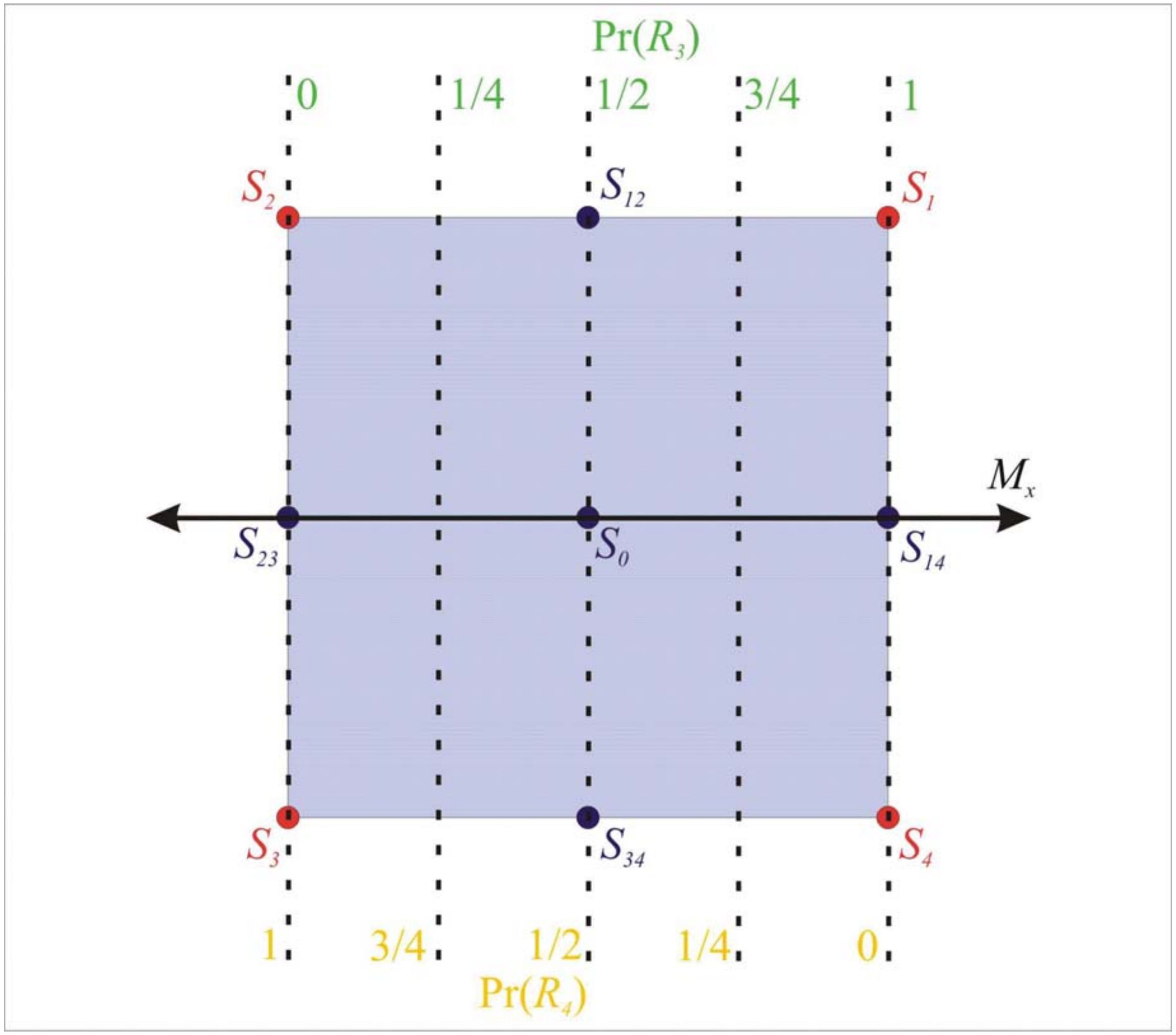


- The set of states can be represented as a square (convex properties)

- Four pure states (cannot be statistically simulated).

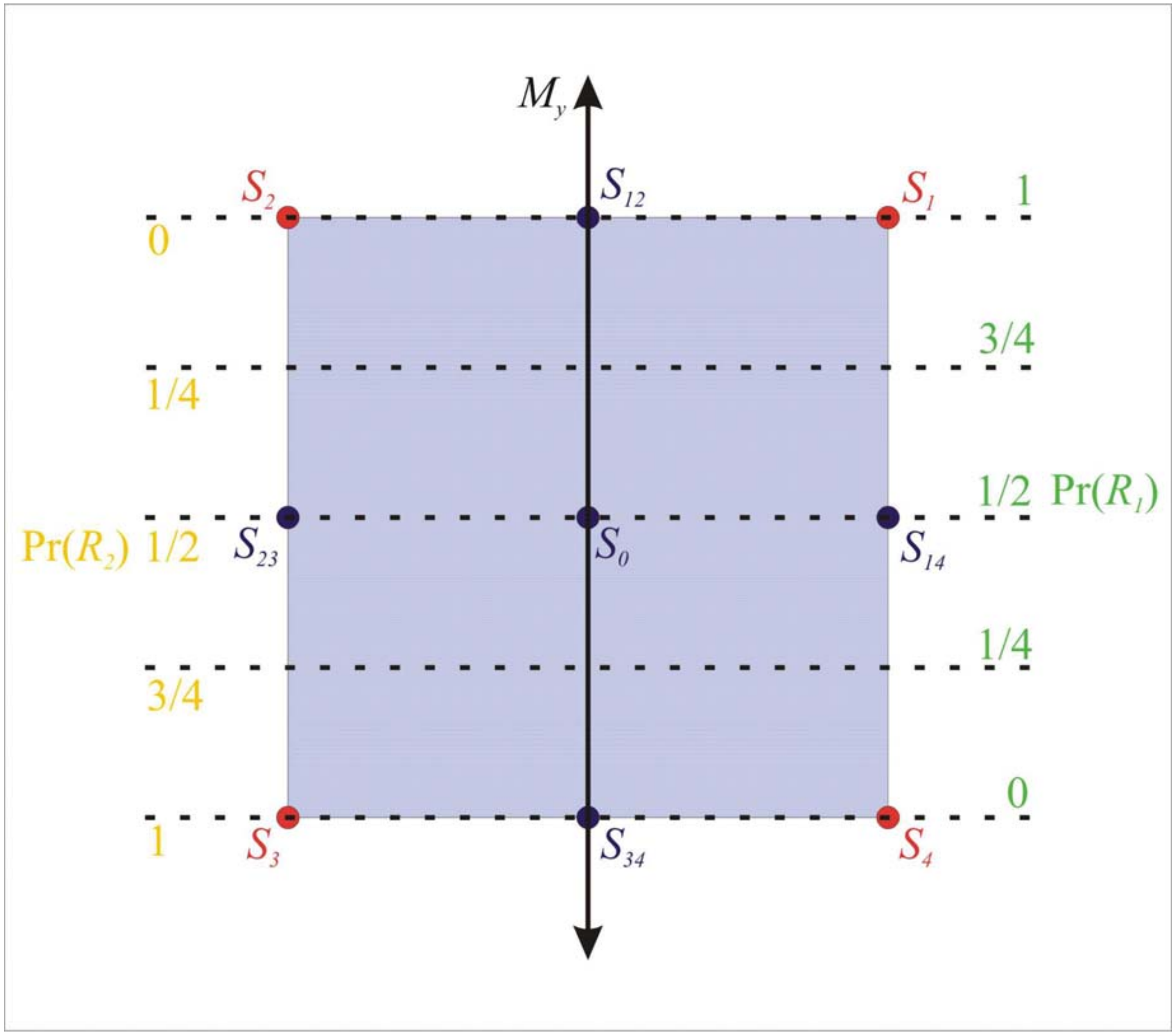
- Some mixed states can be realised in more than one way (statistically non-classical).

Toy model: square representation



- Measurements represented by isoprobability lines.
- Two maximal measurements.
- Pure states cannot be distinguished all ”in one shot”.

Toy model: square representation



- Measurements represented by isoprobability lines.
- Two maximal measurements.
- Pure states cannot be distinguished all "in one shot".

Blankenbecler & Partovi's entropy definition applied to the toy model

Consider a state

S_{12}

Check the (Shannon) entropies for all (pure)measurements upon that state

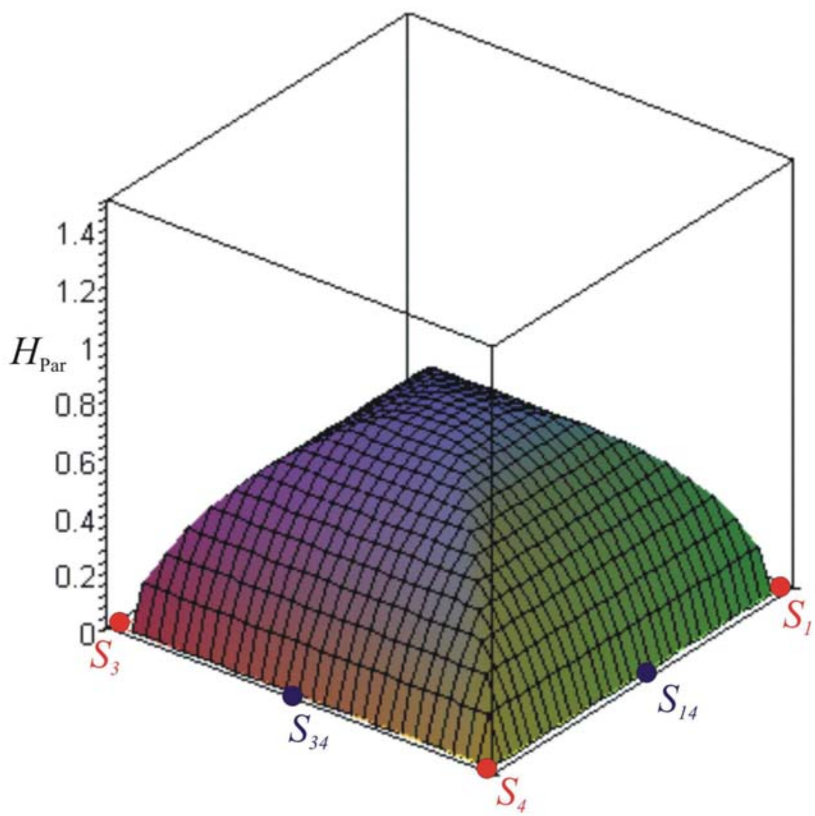
$$H(S_{12}, M_x) = \ln 2 \qquad H(S_{12}, M_y) = 0$$

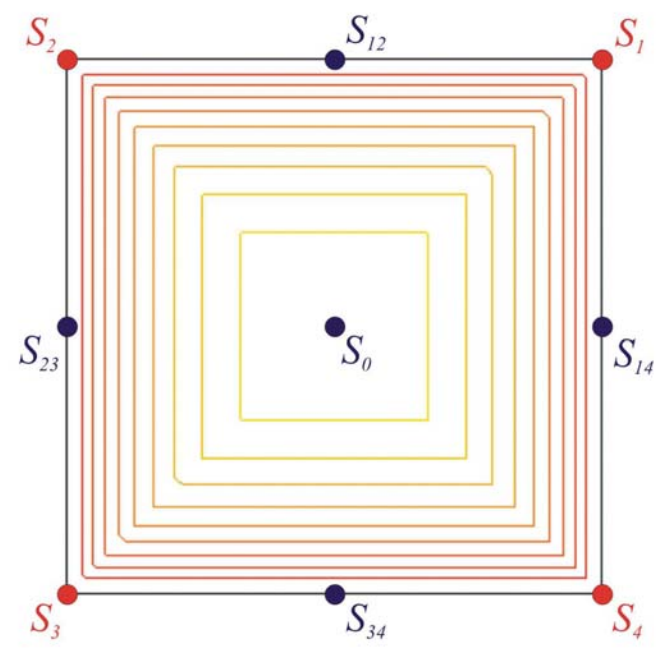
		S_{12}
M_y	$P(R_1)$	1
	$P(R_2)$	0
M_x	$P(R_3)$	1/2
	$P(R_4)$	1/2

Select the least of these entropies as the entropy of that state

$$H_{\text{Par}}(S_{12}) = 0$$

Blankenbecler & Partovi's entropy definition applied to the toy model



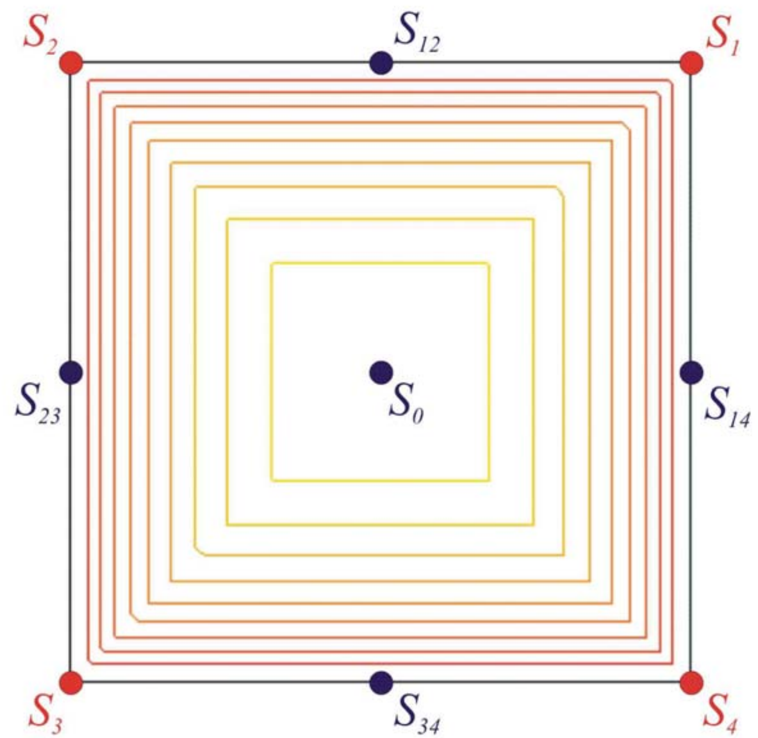


$$H_{\text{Par}}(S) = \min[H(x,1\text{-}x),\ H(y,1\text{-}y)]$$

Blankenbecler & Partovi's entropy definition applied to the toy model

Problem:

the entropy which arises from Blankenbecler & Partovi's definition is not *convex* → it cannot be used in a constrained maximum-entropy principle



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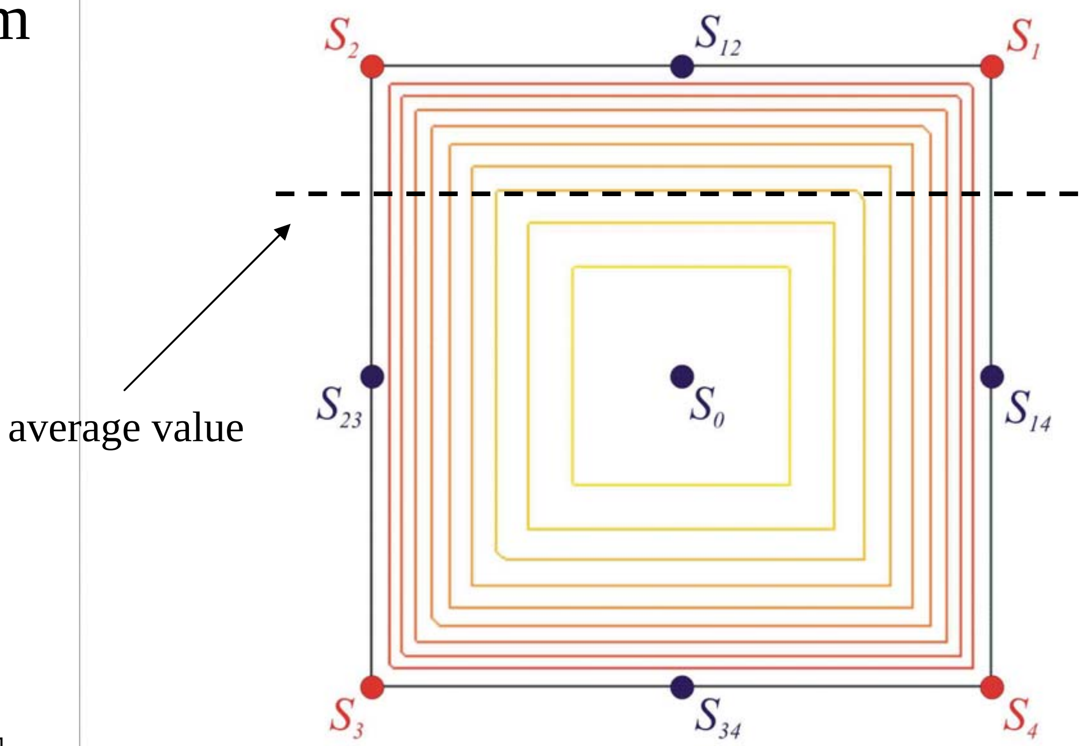

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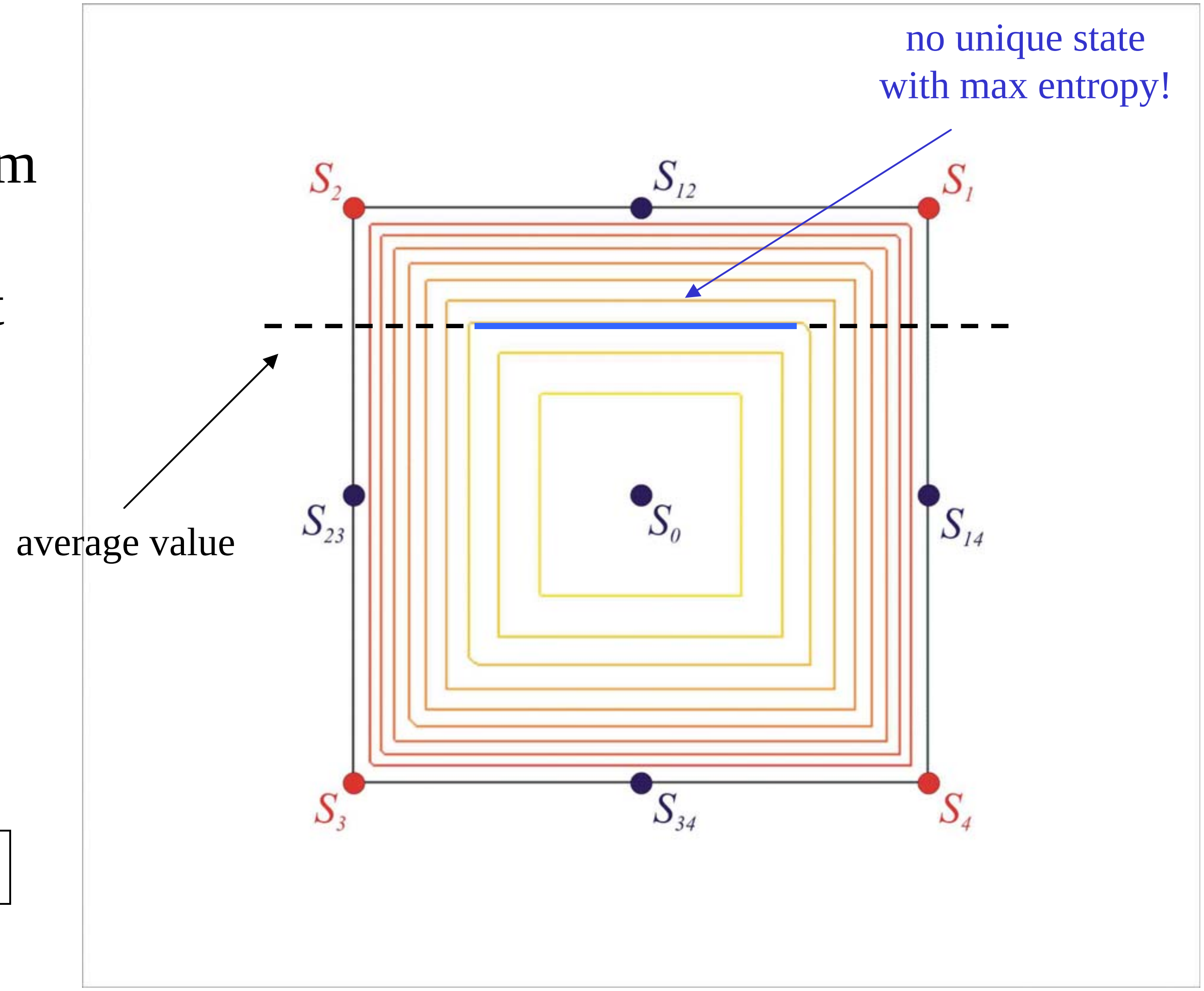

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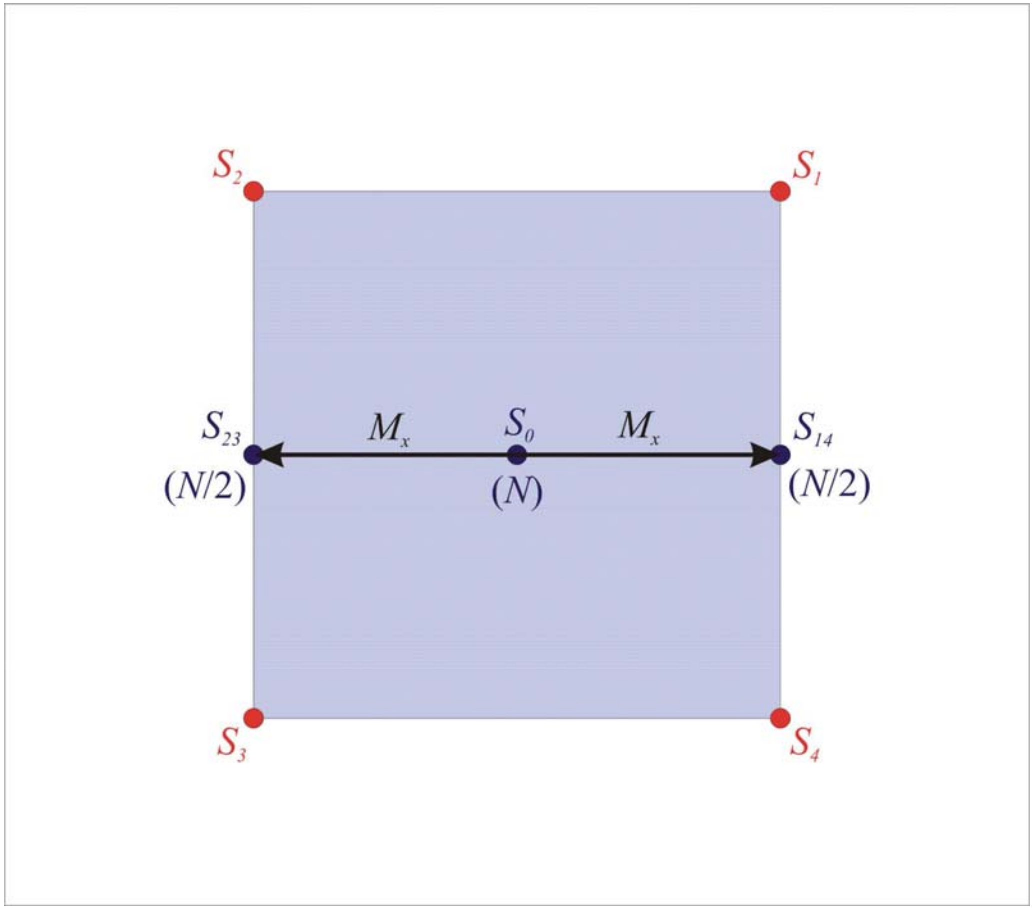
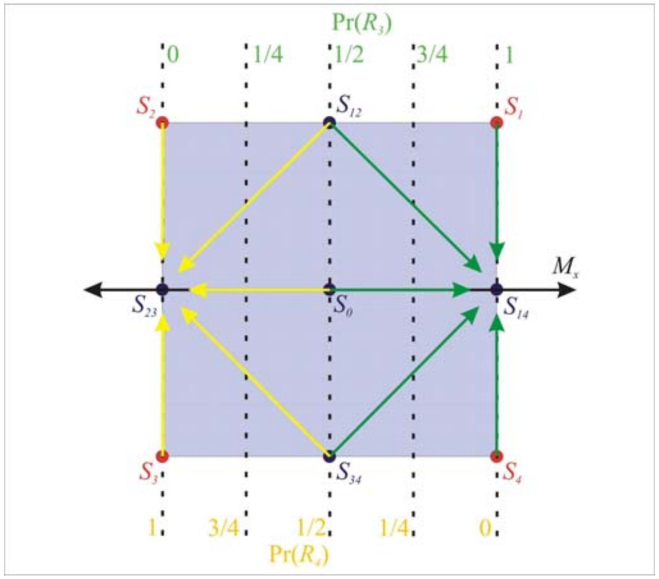
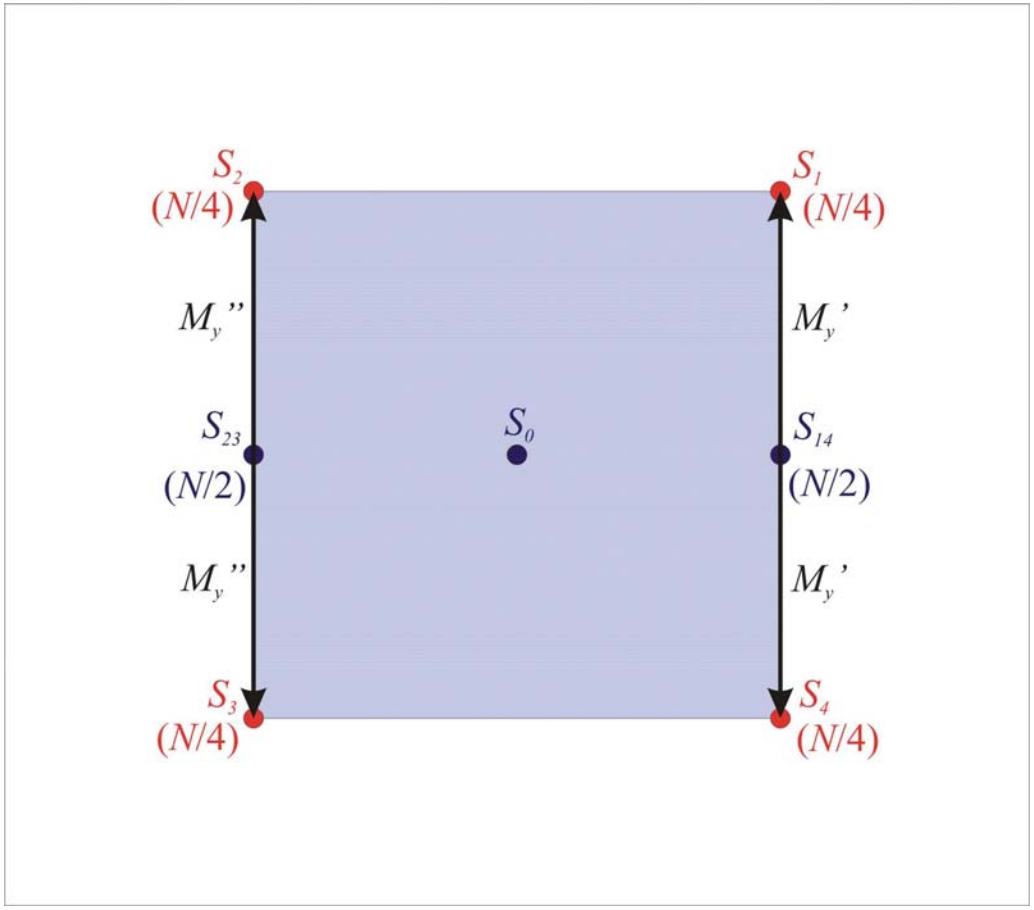
von Neumann's entropy definition applied to the toy model

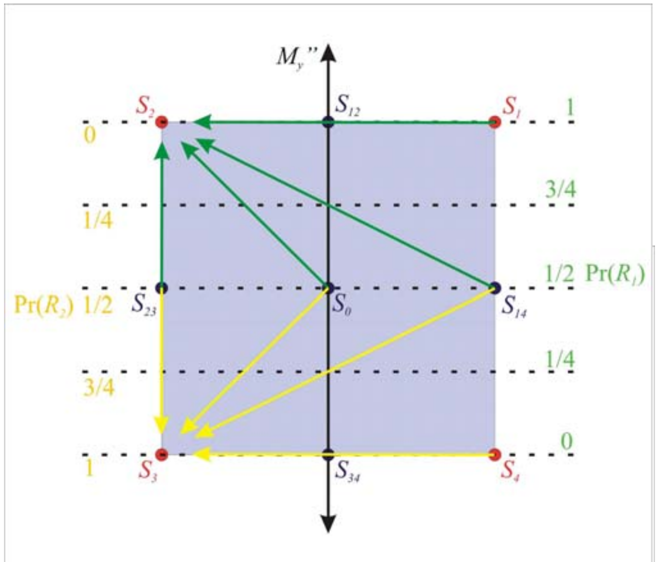


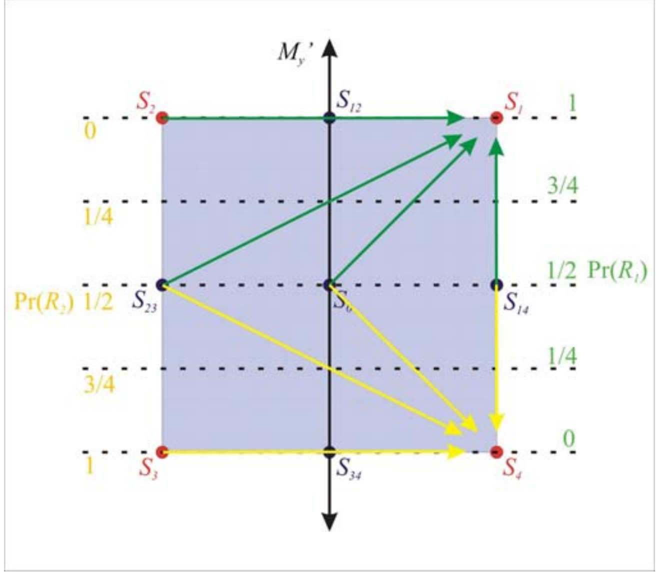


Implementation of the measurements

von Neumann's entropy definition applied to the toy model







Implementation of the measurements

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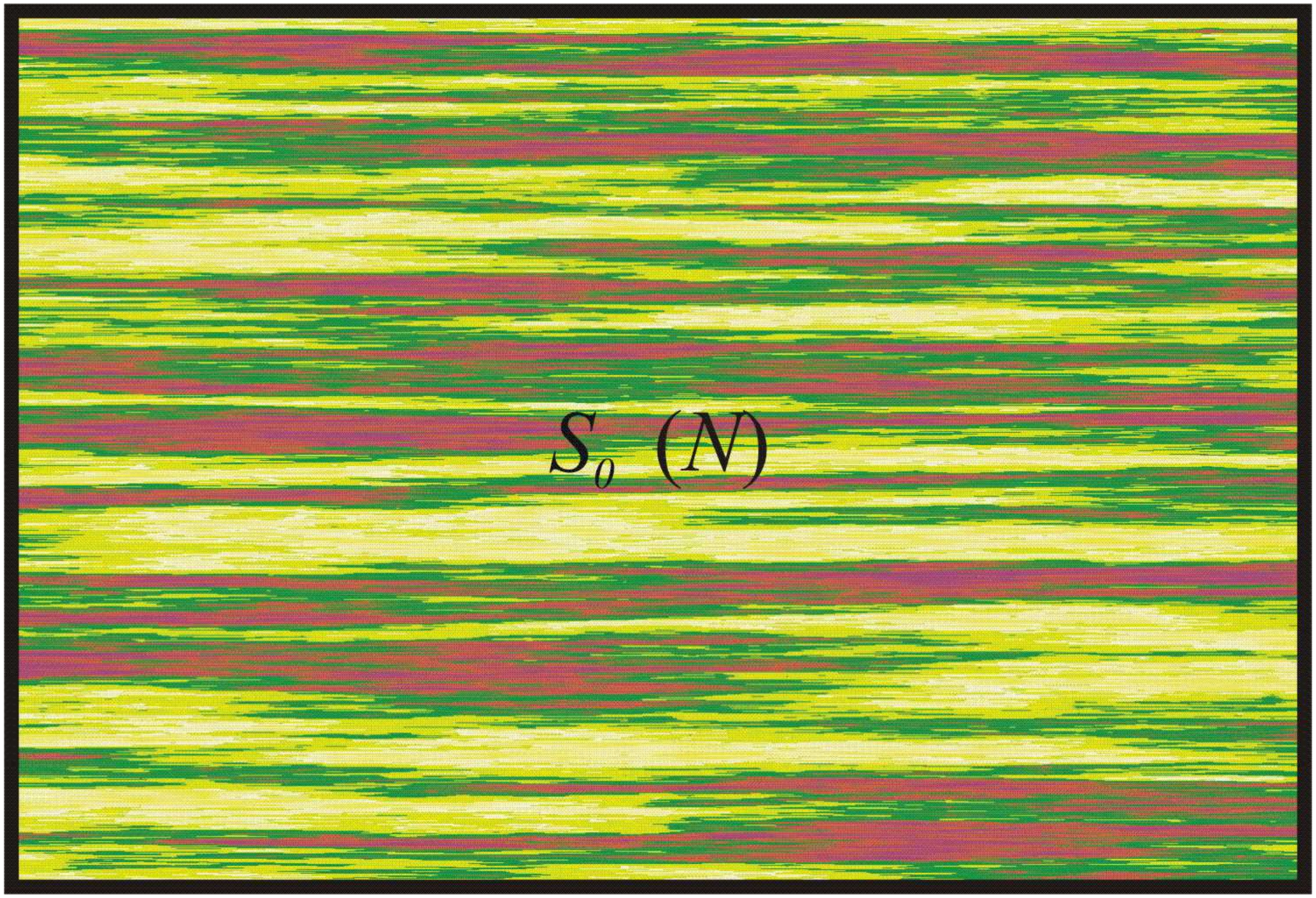
Example: entropy for the mixed state S_0



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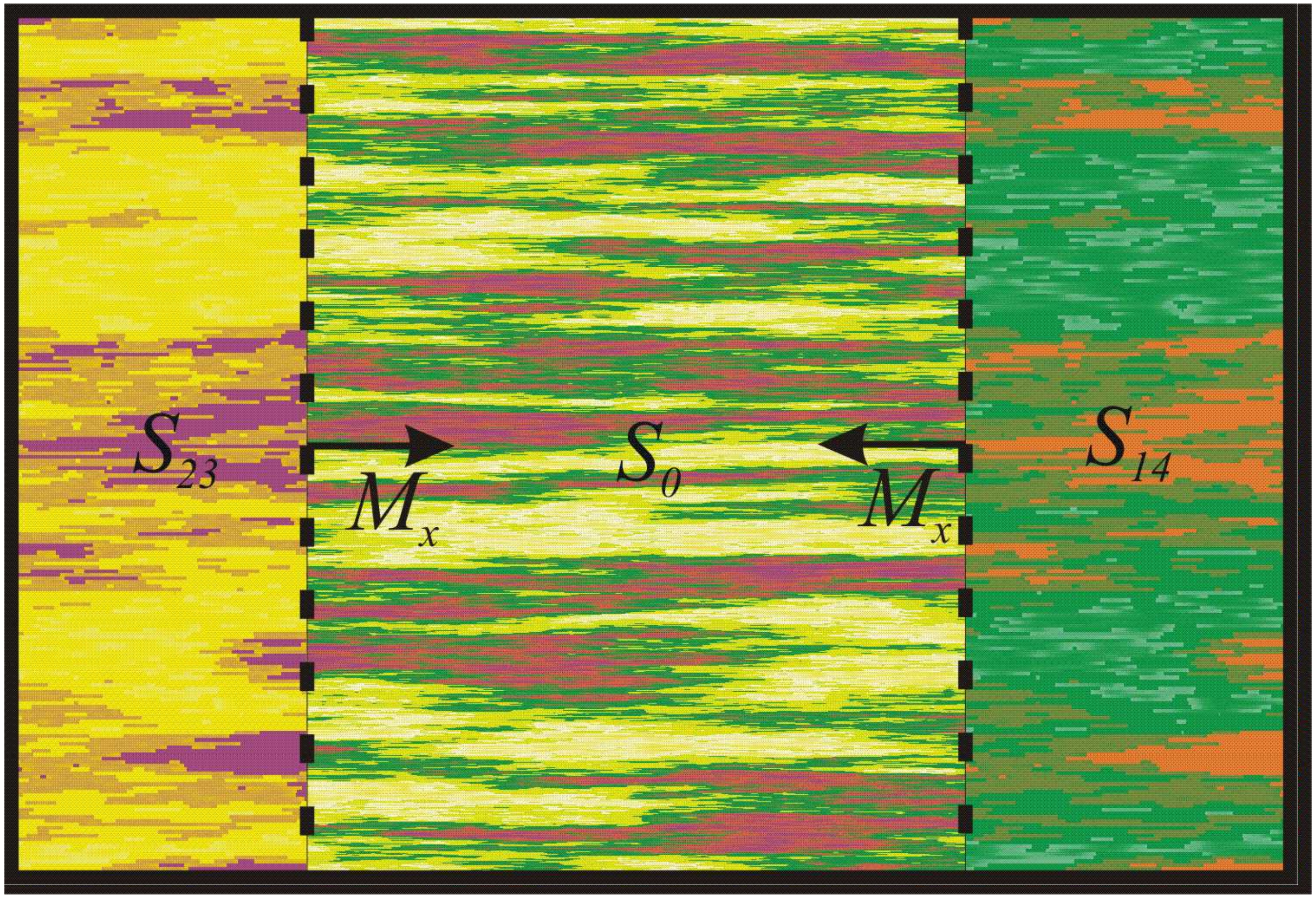

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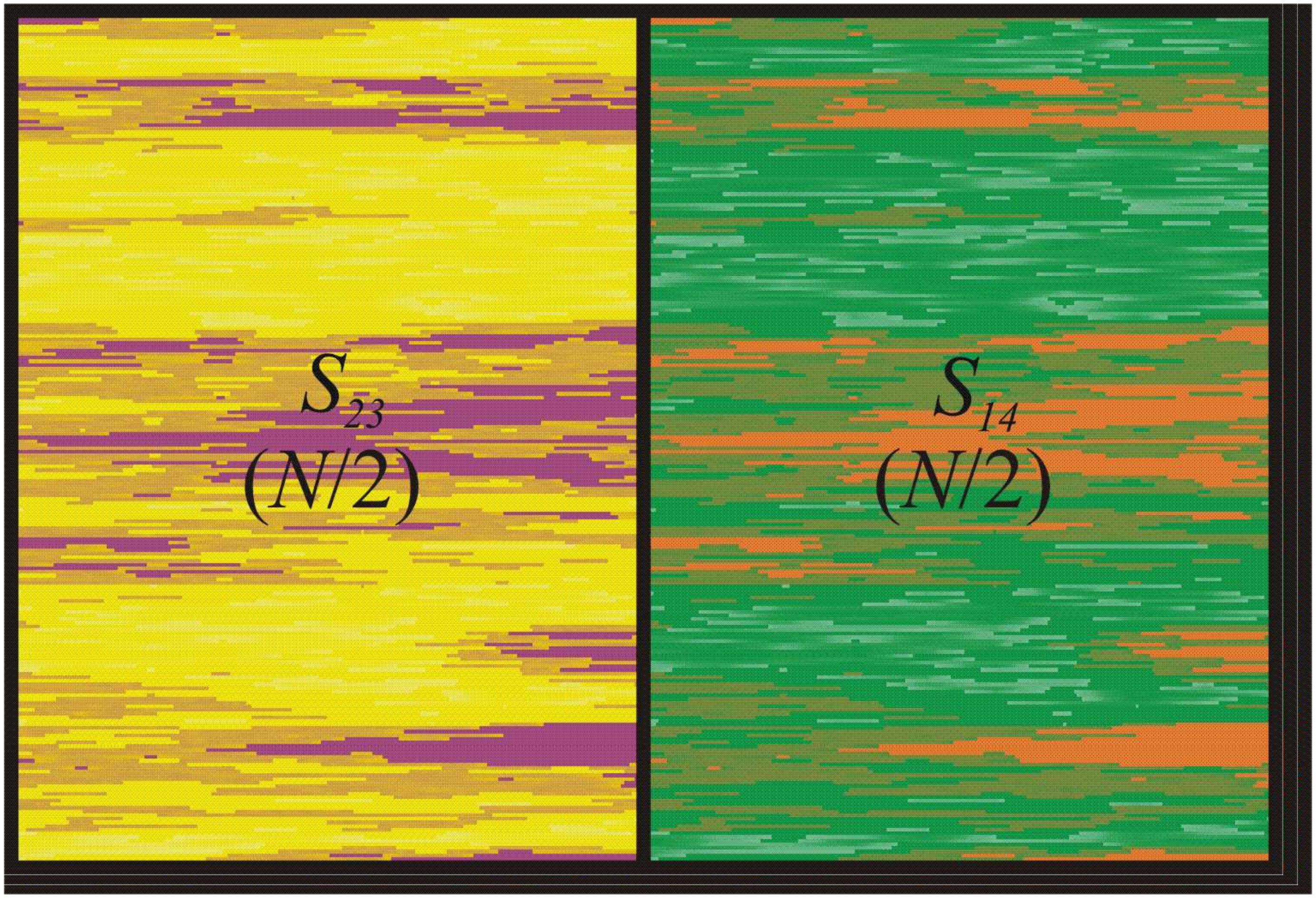

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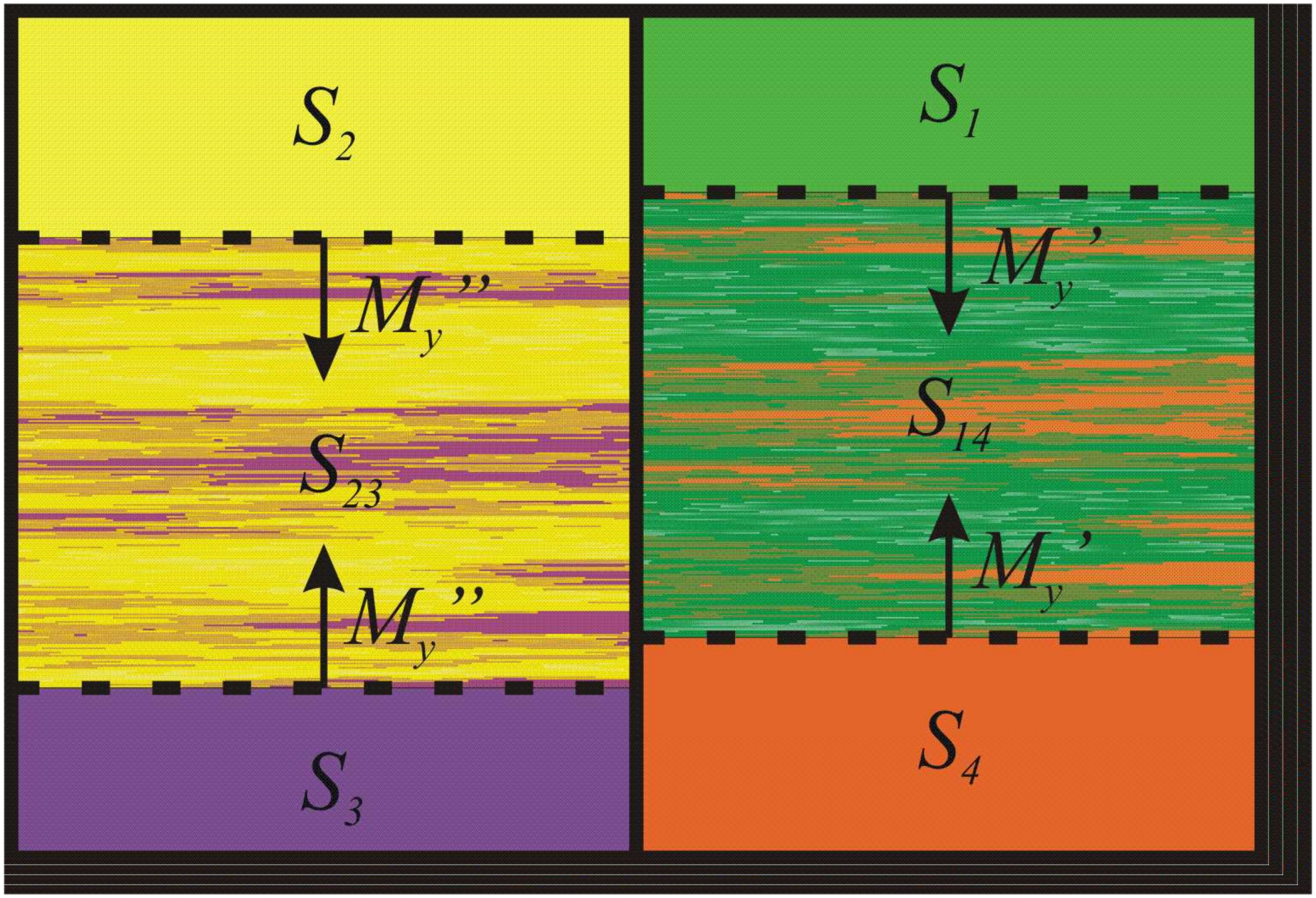
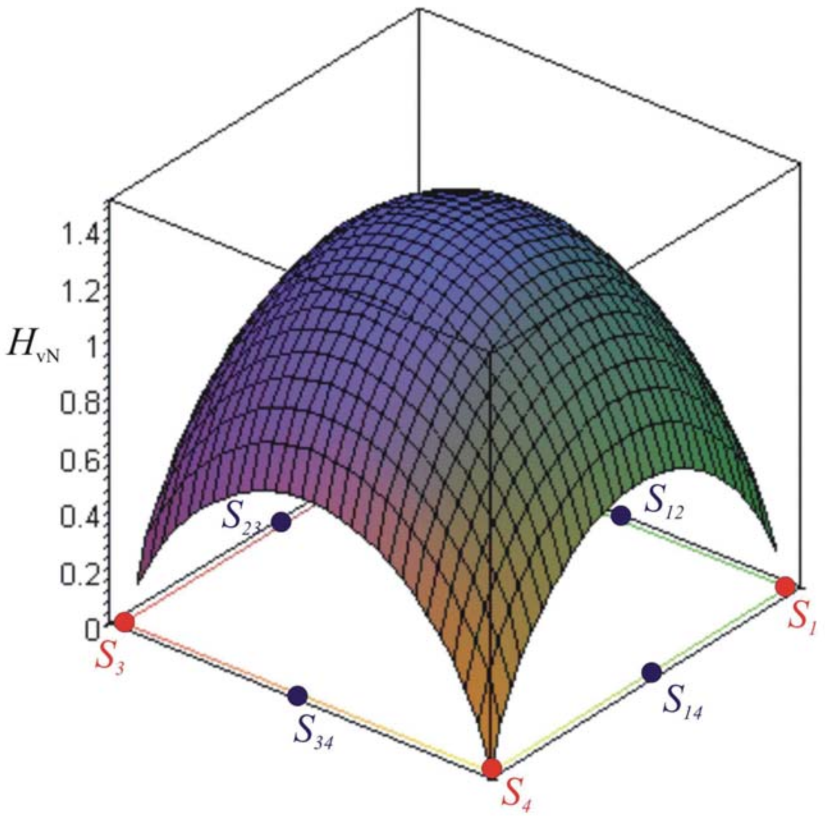


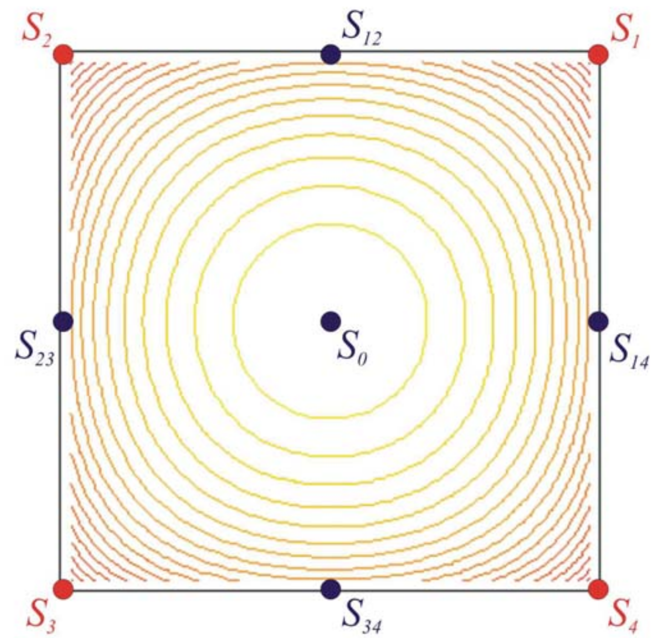
Example: entropy for the mixed state S_0

Example: entropy for the mixed state S_0

$$W = -(2\times N/2 + 4\times N/4)\ T \ln(1/2) = Q \quad \rightarrow \quad H_{\mathrm{vN}}(S_0) = \ln(4)$$

von Neumann's entropy definition applied to the toy model





$$H_{vN}(S) = H(x,1\text{-}x) + H(y,1\text{-}y)$$

von Neumann's entropy definition applied to the toy model

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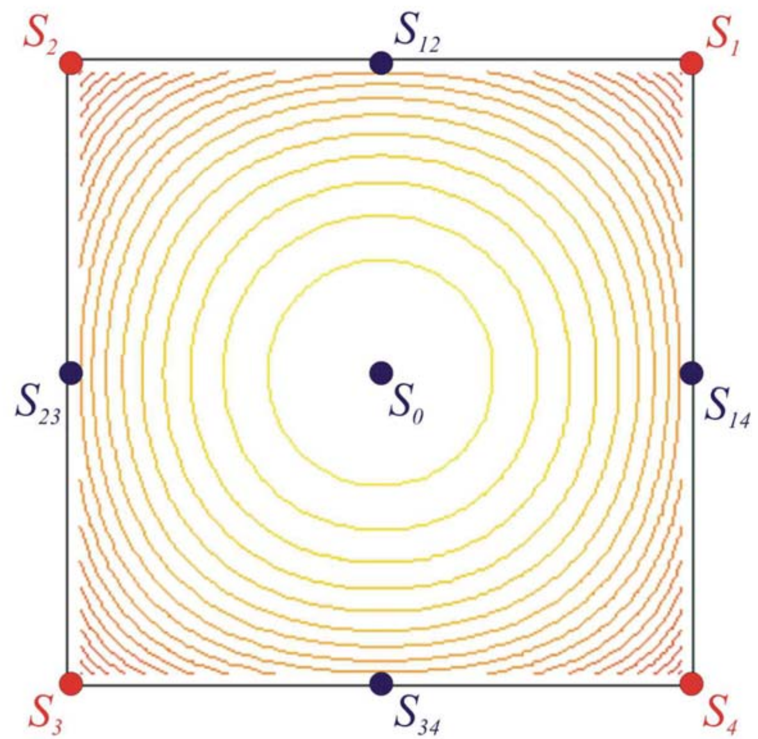
The entropy which arises from von Neumann's definition is convex → it can be used in a constrained maximum-entropy principle



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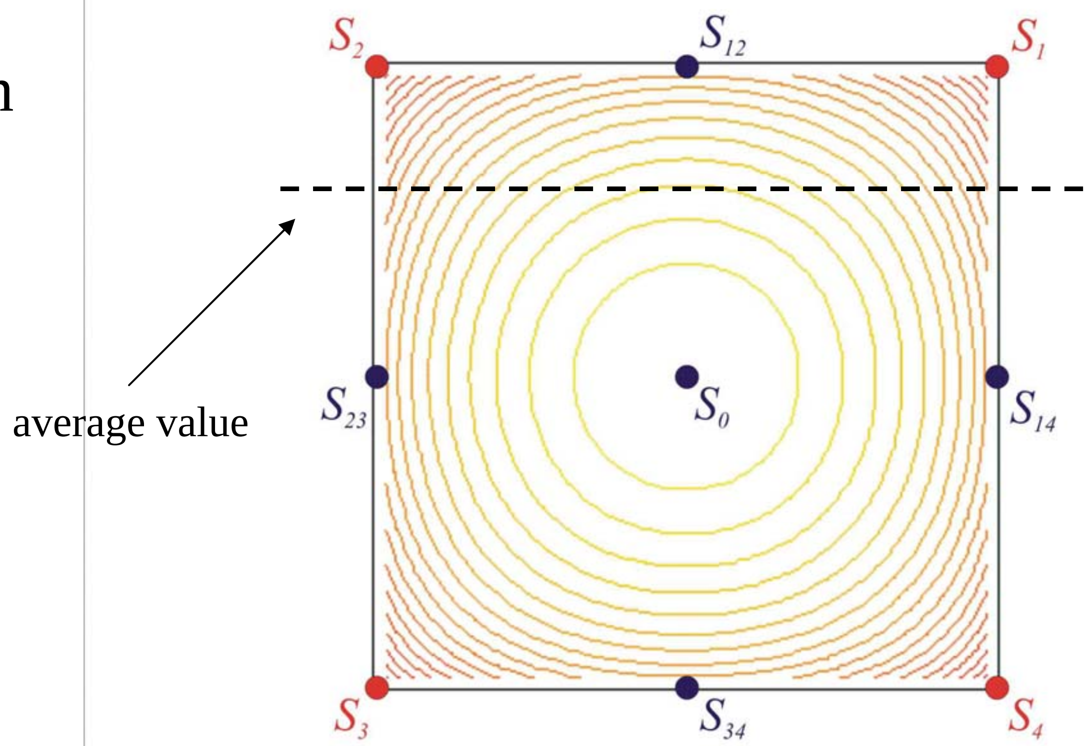

$$H_{vN}(S) = H(x,1-x) + H(y,1-y)$$

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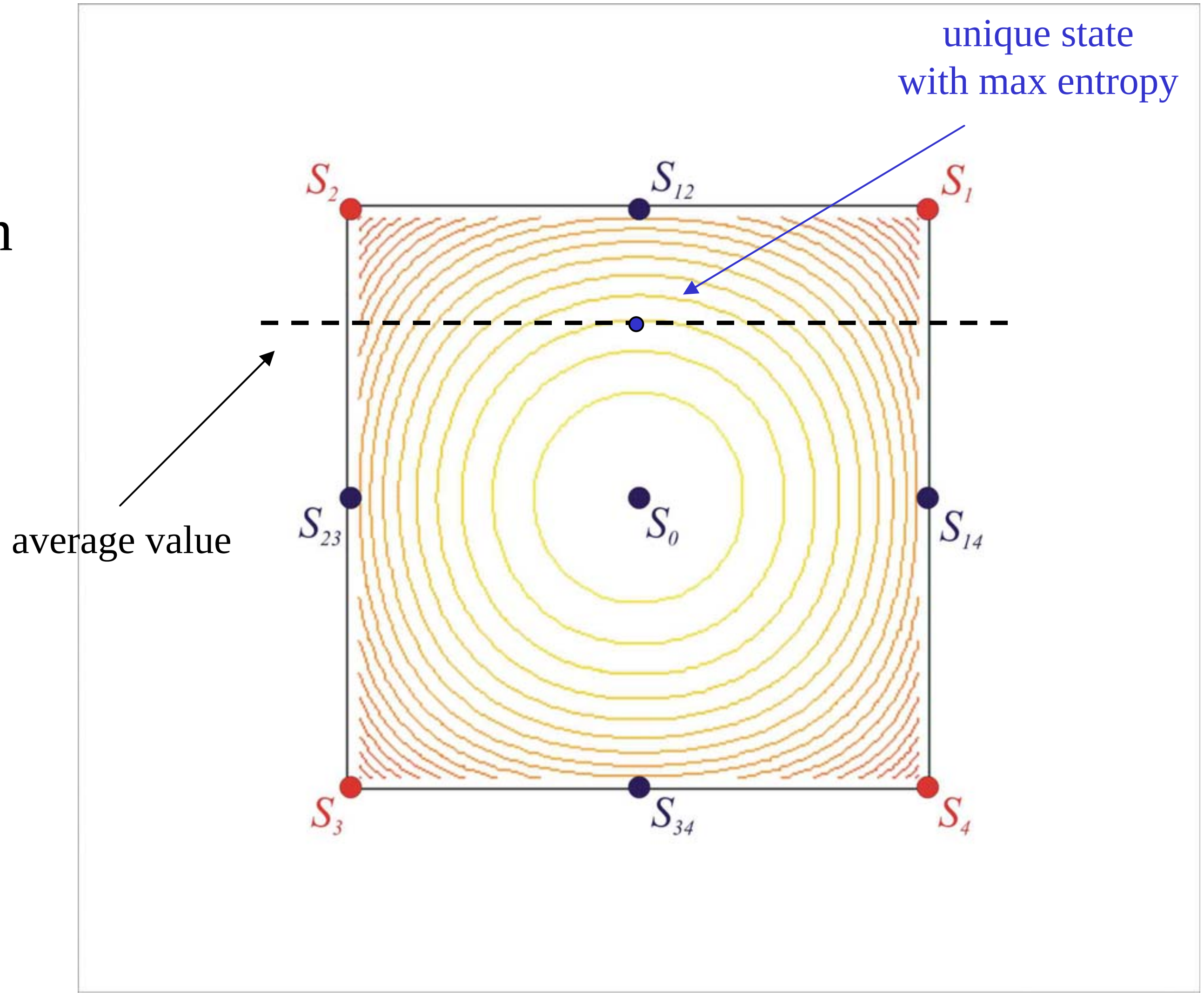

$$H_{vN}(S) = H(x,1-x) + H(y,1-y)$$

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Conclusions and further questions

While the BP and vN definitions for the state entropy agree for quantum systems, they do not agree for more general statistical models

The BP definition does not lead in general to a convex entropy formula

Does the vN definition always lead to a convex entropy formula?

Is there a natural, universal entropy formula for any statistical system?
It should be derived from logical/probability-theoretical principles.
(Particular cases: W. Ochs, Rep. Math. Phys. **8** (1975) 109; I. Csiszár, Ann. Stat. **19** (1991) 2032)